Final exam

Good luck Viel Erfolg 好好儿考啊 시험 잘봐 Semoga sukses амжилт хүсье
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Exercise 1. (14 Points) We define the following matrix

$$A = \begin{pmatrix} 0 & 0 & 2 \\ -3 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}.$$

i) Compute the determinant of A.

ii) Find all eigenvalues of A. For each eigenvalue λ determine a basis of the eigenspace $E_\lambda(A)$.

iii) Is A diagonalizable and/or invertible? Justify your answers.

Exercise 2. (10 Points) Decide if the following statements are true or false. Justify your answer by giving a short explanation.

1) The set $U = \{f \in C^\infty(\mathbb{R},\mathbb{R}) \mid f'' = 2f\}$ is a subspace of $C^\infty(\mathbb{R},\mathbb{R})$.

2) The set $U = \left\{A \in \mathbb{R}^{2\times 2} \mid \det(A) \neq 0\right\}$ is a subspace of $\mathbb{R}^{2\times 2}$.

3) If (b_1, b_2) is a basis of a vector space V, then $(b_1 + b_2, -b_2 - b_1)$ is also a basis of V.

4) If $A \in \mathbb{R}^{n\times n}$ is not invertible then A has at least one eigenvalue.

5) There exists a linear map $F : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies $F \circ F = F$ and which has eigenvalue -1.

Exercise 3. (8 Points) Give an example of

i) a matrix $A \in \mathbb{R}^{3\times 3}$ which is not diagonalizable and not invertible.

ii) a linear map $F : \mathcal{P}_1 \to \mathcal{P}_1$ with $\det(F) = 2$.

iii) a symmetric and orthogonal matrix $B \in \mathbb{R}^{2\times 2}$ which is not the identity matrix.

Justify your examples.

Exercise 4. (10 Points) Let $x_n \in \mathbb{R}^2$ be for $n \geq 0$ be defined by

$$x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad x_{n+1} = M x_n\,, \qquad \text{where} \qquad M = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}.$$

Determine an explicit formula for x_n.

Exercise 5. (8 Points) Find all solutions to the following differential equation

$$f''' - 2f'' + f' = 2020\,.$$

After finishing this exam please send your solution as one pdf file to
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