Tutorial 4: Linear maps – Solutions

Tuesday, 27th October, 2026

Exercise 1. Which of the following functions are linear maps?

$$f_1\colon \mathbb{R}\to\mathbb{R}, \quad x\mapsto \sin(x), \qquad f_2\colon \mathbb{R}\to\mathbb{R}, \quad x\mapsto x^2+1,$$
$$f_3\colon \mathbb{R}^2\to\mathbb{R}^2, \quad \begin{pmatrix}x_1\\x_2\end{pmatrix}\mapsto\begin{pmatrix}8x_1+2x_2\\4x_2\end{pmatrix}, \qquad f_4\colon \mathbb{R}^2\to\mathbb{R}^2, \quad \begin{pmatrix}x_1\\x_2\end{pmatrix}\mapsto\begin{pmatrix}x_1+2x_2\\x_1^2+x_2\end{pmatrix}.$$

Solution.

- f_1 is not linear: $f_1(2\cdot\frac{\pi}{2})=\sin(\pi)=0$, whereas $2f_1(\frac{\pi}{2})=2$.
- f_2 is not linear because $f_2(0)=1\neq 0$. Every linear map sends 0 to 0.
- f_3 is linear because it is given by multiplication by a matrix:
$$f_3\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}8&2\\0&4\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}.$$
Hence $[f_3]=\begin{pmatrix}8&2\\0&4\end{pmatrix}$.
- f_4 is not linear: with $u=\begin{pmatrix}1\\0\end{pmatrix}$,
$$f_4(2u)=\begin{pmatrix}2\\4\end{pmatrix}\neq\begin{pmatrix}2\\2\end{pmatrix}=2f_4(u).$$

Exercise 2. Show that there exists a unique linear map $G\colon\mathbb{R}^2\to\mathbb{R}^3$ with

$$G\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}, \qquad G\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}-1\\1\\-1\end{pmatrix}.$$

What is $G(x)$ for an arbitrary $x=\begin{pmatrix}x_1\\x_2\end{pmatrix}\in\mathbb{R}^2$? Determine the matrix of G.

Solution. Write $v_1=\begin{pmatrix}1\\1\end{pmatrix}$ and $v_2=\begin{pmatrix}1\\2\end{pmatrix}$. Since

$$e_1=2v_1-v_2, \qquad e_2=v_2-v_1,$$

linearity forces

$$G(e_1)=2\begin{pmatrix}1\\0\\1\end{pmatrix}-\begin{pmatrix}-1\\1\\-1\end{pmatrix}=\begin{pmatrix}3\\-1\\3\end{pmatrix}, \qquad G(e_2)=\begin{pmatrix}-1\\1\\-1\end{pmatrix}-\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}-2\\1\\-2\end{pmatrix}.$$

The columns of the matrix of G are therefore forced to be these vectors. Thus the only possible matrix is

$$[G]=\begin{pmatrix}3&-2\\-1&1\\3&-2\end{pmatrix}, \qquad G\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}3x_1-2x_2\\-x_1+x_2\\3x_1-2x_2\end{pmatrix}.$$

Conversely, the map $G(x)=[G]x$ is linear, and

$$[G]\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}, \qquad [G]\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}-1\\1\\-1\end{pmatrix}.$$

It therefore has the required values, proving existence as well as uniqueness.