Exercise Sheet 5: Geometry and matrix multiplication Solutions

Discussed in Tutorial 5 on Tuesday, 10th November, 2026

Exercise 1. Let

$$u = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \qquad v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad w = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

be vectors in $\mathbb{R}^3$.

(a) Calculate $u \bullet v$, $u \bullet w$, $v \bullet w$ and $\|u\|$, $\|v\|$, $\|w\|$. Which pairs of these vectors are orthogonal?

(b) Determine the angle $\alpha \in [0, \pi]$ between u and v, and the angle $\beta \in [0, \pi]$ between v and w.

(c) Determine all vectors $z \in \mathbb{R}^3$ that are orthogonal to both u and v.

Solution. (a) By the definitions of the dot product and norm,

$$\begin{aligned} u \bullet v &= 2 \cdot 1 + 1 \cdot 1 + (-2) \cdot 0 = 3, \\ u \bullet w &= 2 \cdot (-1) + 1 \cdot 4 + (-2) \cdot 1 = 0, \\ v \bullet w &= 1 \cdot (-1) + 1 \cdot 4 + 0 \cdot 1 = 3, \end{aligned}$$

and

$$\|u\| = \sqrt{4+1+4} = 3, \qquad \|v\| = \sqrt{1+1+0} = \sqrt{2}, \qquad \|w\| = \sqrt{1+16+1} = 3\sqrt{2}.$$

Thus only the pair u, w is orthogonal.

(b) For nonzero vectors, the dot product gives the cosine of the angle:

$$\cos(\alpha) = \frac{u \bullet v}{\|u\| \, \|v\|} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}, \qquad \cos(\beta) = \frac{v \bullet w}{\|v\| \, \|w\|} = \frac{3}{\sqrt{2} \cdot 3\sqrt{2}} = \frac{1}{2}.$$

Since both angles lie in $[0, \pi]$, we obtain

$$\boxed{\alpha = \frac{\pi}{4}, \qquad \beta = \frac{\pi}{3}.}$$

(c) Write $z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$. The two orthogonality conditions are

$$\begin{cases} 2z_1 + z_2 - 2z_3 = 0, \\ z_1 + z_2 = 0. \end{cases}$$

The second equation gives $z_2 = -z_1$. Substituting into the first gives $z_1 - 2z_3 = 0$, so $z_1 = 2z_3$ and $z_2 = -2z_3$. Taking $z_3 = t$, all solutions are

$$\boxed{z = t \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \qquad t \in \mathbb{R}.}$$

Conversely, every such vector satisfies both conditions, since

$$u \bullet z = t(4 - 2 - 2) = 0, \qquad v \bullet z = t(2 - 2 + 0) = 0.$$

These vectors form a line through $\mathbf{0}$ in $\mathbb{R}^3$.

Exercise 2. Let $u = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $x = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, and let $P_u : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto the line $\{tu \mid t \in \mathbb{R}\}$.

(a) Determine the matrix $[P_u]$ and calculate $P_u(x)$ and $x - P_u(x)$. Check that $x - P_u(x)$ is orthogonal to u.

(b) Draw u, x, $P_u(x)$ and $x - P_u(x)$ in one picture, together with the line $\{tu \mid t \in \mathbb{R}\}$.

(c) More generally, let $v \in \mathbb{R}^n$ with $v \neq \mathbf{0}$. Show that for every $y \in \mathbb{R}^n$,

$$v \bullet \big(y - P_v(y)\big) = 0, \qquad P_v\big(P_v(y)\big) = P_v(y).$$

Explain the geometric meaning of these two identities.

Solution. (a) Since $u \bullet u = 2^2 + (-3)^2 = 13$, the columns of $[P_u]$ are

$$P_u(e_1) = \frac{2}{13}\begin{pmatrix} 2 \\ -3 \end{pmatrix}, \qquad P_u(e_2) = -\frac{3}{13}\begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

Hence

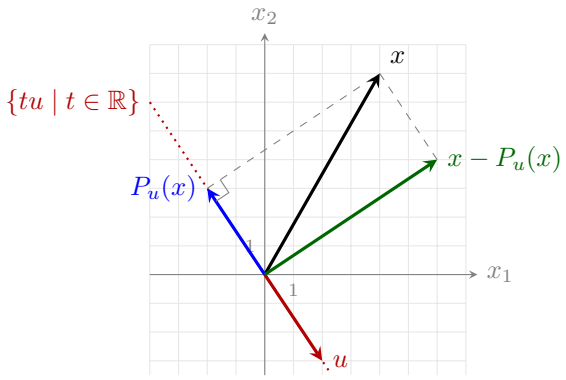
$$\boxed{[P_u] = \frac{1}{13}\begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix}.}$$

As $u \bullet x = 2 \cdot 4 + (-3) \cdot 7 = -13$, we get

$$P_u(x) = \frac{-13}{13}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \boxed{\begin{pmatrix} -2 \\ 3 \end{pmatrix}}, \qquad x - P_u(x) = \begin{pmatrix} 4 \\ 7 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \boxed{\begin{pmatrix} 6 \\ 4 \end{pmatrix}}.$$

Indeed, $u \bullet \big(x - P_u(x)\big) = 2 \cdot 6 + (-3) \cdot 4 = 0$.

(b) The dashed segment from $P_u(x)$ to x is a translated copy of $x - P_u(x)$.



(c) Since $v \neq \mathbf{0}$, we have $v \bullet v > 0$. Using the dot product properties,

$$v \bullet P_v(y) = v \bullet \left(\frac{v \bullet y}{v \bullet v} v\right) = \frac{v \bullet y}{v \bullet v}(v \bullet v) = v \bullet y.$$

Thus $v \bullet \big(y - P_v(y)\big) = v \bullet y - v \bullet P_v(y) = 0$, and

$$P_v\big(P_v(y)\big) = \frac{v \bullet P_v(y)}{v \bullet v} v = \frac{v \bullet y}{v \bullet v} v = P_v(y).$$

The first identity says that the difference between a vector and its projection is perpendicular to the line. The second says that projecting a vector already on the line leaves it unchanged.

Exercise 3.

(a) Let

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ -2 & 0 \\ 1 & -1 \end{pmatrix}.$$

Which of the products $A \cdot B$, $B \cdot A$, $A \cdot A$ and $B \cdot B$ are defined? For each product that is defined, give its size and calculate it.

(b) For u and x from Exercise 2, determine the matrices

$$[P_u \circ \mathrm{rot}_{\frac{\pi}{2}}], \qquad [\mathrm{rot}_{\frac{\pi}{2}} \circ P_u].$$

Use them to calculate $(P_u \circ \mathrm{rot}_{\frac{\pi}{2}})(x)$ and $(\mathrm{rot}_{\frac{\pi}{2}} \circ P_u)(x)$. Are the two compositions the same linear map?

Solution. (a) The matrix A has size 2×3, and B has size 3×2. The number of columns of the left factor must equal the number of rows of the right factor. Thus $A \cdot B$ and $B \cdot A$ are defined, of sizes 2×2 and 3×3, respectively. Neither $A \cdot A$ nor $B \cdot B$ is defined.

Multiplying each row of the left matrix with each column of the right matrix gives

$$A \cdot B = \begin{pmatrix} 2+2+1 & 4+0-1 \\ 0-2+3 & 0+0-3 \end{pmatrix} = \boxed{\begin{pmatrix} 5 & 3 \\ 1 & -3 \end{pmatrix}},$$

and

$$B \cdot A = \begin{pmatrix} 2+0 & -1+2 & 1+6 \\ -4+0 & 2+0 & -2+0 \\ 2+0 & -1-1 & 1-3 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 1 & 7 \\ -4 & 2 & -2 \\ 2 & -2 & -2 \end{pmatrix}}.$$

(b) Write $M = [P_u]$ and $R = [\mathrm{rot}_{\frac{\pi}{2}}]$. From Exercise 2 and the rotation formula,

$$M = \frac{1}{13}\begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The matrix of a composition is the product of the matrices in the same order:

$$[P_u \circ \mathrm{rot}_{\frac{\pi}{2}}] = M \cdot R = \frac{1}{13}\begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \boxed{\frac{1}{13}\begin{pmatrix} -6 & -4 \\ 9 & 6 \end{pmatrix}},$$

$$[\mathrm{rot}_{\frac{\pi}{2}} \circ P_u] = R \cdot M = \frac{1}{13}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix} = \boxed{\frac{1}{13}\begin{pmatrix} 6 & -9 \\ 4 & -6 \end{pmatrix}}.$$

For $x = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, this gives

$$(P_u \circ \mathrm{rot}_{\frac{\pi}{2}})(x) = \frac{1}{13}\begin{pmatrix} -6 \cdot 4 - 4 \cdot 7 \\ 9 \cdot 4 + 6 \cdot 7 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} -52 \\ 78 \end{pmatrix} = \boxed{\begin{pmatrix} -4 \\ 6 \end{pmatrix}},$$

$$(\mathrm{rot}_{\frac{\pi}{2}} \circ P_u)(x) = \frac{1}{13}\begin{pmatrix} 6 \cdot 4 - 9 \cdot 7 \\ 4 \cdot 4 - 6 \cdot 7 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} -39 \\ -26 \end{pmatrix} = \boxed{\begin{pmatrix} -3 \\ -2 \end{pmatrix}}.$$

The outputs are different, so the two compositions are different linear maps. In $P_u \circ \mathrm{rot}_{\frac{\pi}{2}}$ we first rotate and then project; in $\mathrm{rot}_{\frac{\pi}{2}} \circ P_u$ we first project and then rotate. This also gives an example of $M \cdot R \neq R \cdot M$.