Exercise Sheet 5: Geometry and matrix multiplication

Preparation for Tutorial 5 & Quiz 5 on Tuesday, 10th November, 2026

You do not need to submit solutions to this exercise sheet. It is just for practicing for the quiz.

Solutions can be found on the homepage: `https://www.henrikbachmann.com/la1_2026.html`.

Exercise 1. Let

$$u = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \qquad v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad w = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$$

be vectors in $\mathbb{R}^3$.

(a) Calculate $u \bullet v$, $u \bullet w$, $v \bullet w$ and $\|u\|$, $\|v\|$, $\|w\|$. Which pairs of these vectors are orthogonal?

(b) Determine the angle $\alpha \in [0, \pi]$ between u and v, and the angle $\beta \in [0, \pi]$ between v and w.

(c) Determine all vectors $z \in \mathbb{R}^3$ that are orthogonal to both u and v.

Exercise 2. Let $u = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $x = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, and let $P_u : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto the line $\{tu \mid t \in \mathbb{R}\}$.

(a) Determine the matrix $[P_u]$ and calculate $P_u(x)$ and $x - P_u(x)$. Check that $x - P_u(x)$ is orthogonal to u.

(b) Draw u, x, $P_u(x)$ and $x - P_u(x)$ in one picture, together with the line $\{tu \mid t \in \mathbb{R}\}$.

(c) More generally, let $v \in \mathbb{R}^n$ with $v \neq \mathbf{0}$. Show that for every $y \in \mathbb{R}^n$,

$$v \bullet \big(y - P_v(y)\big) = 0, \qquad P_v\big(P_v(y)\big) = P_v(y).$$

Explain the geometric meaning of these two identities.

Exercise 3.

(a) Let

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ -2 & 0 \\ 1 & -1 \end{pmatrix}.$$

Which of the products $A \cdot B$, $B \cdot A$, $A \cdot A$ and $B \cdot B$ are defined? For each product that is defined, give its size and calculate it.

(b) For u and x from Exercise 2, determine the matrices

$$[P_u \circ \mathrm{rot}_{\frac{\pi}{2}}], \qquad [\mathrm{rot}_{\frac{\pi}{2}} \circ P_u].$$

Use them to calculate $(P_u \circ \mathrm{rot}_{\frac{\pi}{2}})(x)$ and $(\mathrm{rot}_{\frac{\pi}{2}} \circ P_u)(x)$. Are the two compositions the same linear map?