Exercise Sheet 4: Linear maps and their matrices
Solutions

Discussed in Tutorial 4 on Tuesday, 27th October, 2026

Exercise 1. Which of the following functions are linear maps? For each function that is linear, determine its matrix. For each function that is not linear, give an explicit counterexample.

$$f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 - x_3 \\ 2x_2 + x_3 \end{pmatrix},$$

$$f_2 : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto 4x - 3,$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto |x_1| + x_2,$$

$$f_4 : \mathbb{R} \longrightarrow \mathbb{R}^2, \quad x \longmapsto \begin{pmatrix} (x+1)^2 - x^2 - 1 \\ 3x \end{pmatrix}.$$

Solution.

f_1: This function is linear, since for every $x \in \mathbb{R}^3$,

$$f_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Every function $x \mapsto Ax$ is linear, and its matrix is A. Thus

$$[f_1] = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \end{pmatrix}.$$

Its three columns are $f_1(e_1), f_1(e_2), f_1(e_3)$, in that order.

f_2: This function is not linear. Take $u = v = 0$. Then

$$f_2(u+v) = f_2(0) = -3, \qquad f_2(u) + f_2(v) = -3 + (-3) = -6.$$

These values differ, so additivity fails.

f_3: This function is not linear. Take $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\lambda = -1$. Then

$$f_3(\lambda u) = f_3 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 1, \qquad \lambda f_3(u) = -f_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1.$$

Thus $f_3(\lambda u) \neq \lambda f_3(u)$. Notice that $f_3(\mathbf{0}) = 0$: sending zero to zero is not enough to make a function linear.

f_4: Simplifying the first entry gives

$$(x+1)^2 - x^2 - 1 = 2x.$$

Consequently $f_4(x) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} x$ for every $x \in \mathbb{R}$, so this function is linear and

$$[f_4] = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

This is a 2×1 matrix: there is one input coordinate and two output coordinates.

Exercise 2.

(a) For $j = 1, 2, 3$, let $e_j \in \mathbb{R}^3$ have j-th entry 1 and all other entries 0. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear map with

$$F(e_1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \qquad F(e_2) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \qquad F(e_3) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Determine the matrix $[F]$ and calculate $F \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$.

(b) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \qquad G \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Calculate $G \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and determine the matrix $[G]$.

(c) Is there a linear map $H : \mathbb{R}^2 \to \mathbb{R}^2$ with the following property? Justify your answer.

$$H \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad H \begin{pmatrix} -3 \\ 6 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}.$$

Solution. (a) The j-th column of $[F]$ is $F(e_j)$. Hence

$$[F] = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & 1 \end{pmatrix}, \qquad F \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 0 \cdot (-2) + 1 \cdot 3 \\ (-1) \cdot 1 + 3 \cdot (-2) + 1 \cdot 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}.$$

Equivalently, use $F(e_1 - 2e_2 + 3e_3) = F(e_1) - 2F(e_2) + 3F(e_3)$.

(b) Since $\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, linearity gives

$$G \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3\,G \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2\,G \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$$

To find the matrix, we need the values at $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The second is given, and $e_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives

$$G(e_1) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - 2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \qquad G(e_2) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Therefore

$$[G] = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}.$$

The prescribed values $G \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $G \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ cannot simply be used as the two columns: the first input is not e_1.

(c) There is no such linear map. Since $\begin{pmatrix} -3 \\ 6 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, linearity would force

$$H \begin{pmatrix} -3 \\ 6 \end{pmatrix} = -3\,H \begin{pmatrix} 1 \\ -2 \end{pmatrix} = -3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \end{pmatrix} \neq \begin{pmatrix} -6 \\ 3 \end{pmatrix}.$$

This contradicts the second prescribed value.