Exercise Sheet 2: Matrices & vectors

Preparation for Tutorial 2 & Quiz 2 on Tuesday, 13th October, 2026

You do not need to submit solutions to this exercise sheet.
It is just for practicing for the quiz.

Solutions can be found on the homepage: `https://www.henrikbachmann.com/la1_2026.html`

Exercise 1. We define the following matrices and vectors:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 \\ -2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 1 \\ 0 & -1 \\ 2 & 2 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 5 \end{pmatrix},$$
$$u = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad w = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}.$$

(i) Decide which of the following expressions are defined. Evaluate them if possible.

$$Av, \quad Au, \quad vA, \quad 3E, \quad A+B, \quad A-2E, \quad Bu, \quad Bv,$$
$$Cu, \quad u+v, \quad v-2w, \quad uv, \quad A+C, \quad B(Bu), \quad C+w, \quad Ew\,.$$

(ii) Let $b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Write the linear system $Ax = b$ explicitly (with the variables x_1, x_2, x_3) and write down its augmented matrix $(A \mid b)$. Compute $\operatorname{rref}(A \mid b)$ and find all solutions $x \in \mathbb{R}^3$ of $Ax = b$. Check that v is one of them.

Exercise 2.

(i) Compute the row-reduced echelon form of each of the following matrices:

$$M_1 = \begin{pmatrix} 2 & -4 \\ -1 & 2 \\ 3 & -6 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & 3 & 4 \\ 1 & -2 & 3 & -1 \end{pmatrix}, \qquad M_3 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

(ii) Each of the matrices M_1, M_2, M_3 is the augmented matrix $(A \mid b)$ of a linear system $Ax = b$: of a linear system in the variable x_1 for M_1, in the variables x_1, x_2, x_3 for M_2 and in the variables x_1, x_2 for M_3. Use (i) to decide which of these three linear systems have no solution, exactly one solution or infinitely many solutions, and find all solutions.

(iii) Let $a \in \mathbb{R}$ and

$$N = \begin{pmatrix} 1 & -2 & 1 \\ 3 & -5 & 1 \\ -2 & 6 & a \end{pmatrix}.$$

Determine $\operatorname{rref}(N)$ for every $a \in \mathbb{R}$. For which $a \in \mathbb{R}$ is $x = \mathbf{0}$ the only solution $x \in \mathbb{R}^3$ of $Nx = \mathbf{0}$? For all other values of a, find all solutions $x \in \mathbb{R}^3$ of $Nx = \mathbf{0}$.

Exercise 3. Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ be a polynomial of degree at most 3 with real coefficients $a_0, a_1, a_2, a_3 \in \mathbb{R}$. For such a polynomial p we define the vector

$$v_p = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^4 .$$

(i) Find a matrix $D \in \mathbb{R}^{4\times 4}$ such that $v_q = Dv_p$ for every such p, where $q(x) = p(x) - p'(x)$. (Here p' denotes the derivative of the polynomial p with respect to x.)

(ii) Compute the row-reduced echelon form of D.

(iii) Find the polynomial s of degree at most 3 with $Dv_s = v_p$, where $p(x) = 1 + x^2 - x^3$.

(iv) For an arbitrary polynomial p of degree at most 3, find the polynomial s of degree at most 3 with $Dv_s = v_p$. Can you express s in terms of p and its derivatives?

Exercise 4. Let $A \in \mathbb{R}^{3\times 3}$ be a matrix.

(i) Explain why $x = \mathbf{0}$ is always a solution of $Ax = \mathbf{0}$ and why $\operatorname{rref}(A \mid \mathbf{0}) = (\operatorname{rref}(A) \mid \mathbf{0})$.

(ii) Show that if every column of $\operatorname{rref}(A)$ contains a pivot element, then $x = \mathbf{0}$ is the only vector $x \in \mathbb{R}^3$ with $Ax = \mathbf{0}$.

(iii) Show that if some column of $\operatorname{rref}(A)$ contains no pivot element, then there exist infinitely many vectors $x \in \mathbb{R}^3$ with $Ax = \mathbf{0}$.

(iv) Let $x, y \in \mathbb{R}^3$ with $Ax = \mathbf{0}$ and $Ay = \mathbf{0}$, and let $\lambda \in \mathbb{R}$. Show that $A(x+y) = \mathbf{0}$ and $A(\lambda x) = \mathbf{0}$.