Exercise Sheet 1: Linear systems – Solutions

Discussed in Tutorial 1 on Tuesday, 6th October, 2026

Notation. The row operations are written as in the lecture: a circled number c next to an equation with an arrow to another equation means that c times the first equation is added to the other one (R1); a circled number without an arrow means that the equation is multiplied with this number (R2); a double arrow means that two equations are exchanged (R3).

Exercise 1. Find all solutions of the following linear systems in the variables x_1, x_2.

(i) $\begin{cases} x_1 + 2x_2 = 4 \\ 3x_1 - x_2 = 5 \end{cases}$ (ii) $\begin{cases} 2x_1 + 3x_2 = 1 \\ 5x_1 + 4x_2 = 13 \end{cases}$ (iii) $\begin{cases} 3x_1 - 6x_2 = 9 \\ -2x_1 + 4x_2 = -6 \end{cases}$

(iv) $\begin{cases} 2x_1 - x_2 = 3 \\ -6x_1 + 3x_2 = 1 \end{cases}$ (v) $\{\ 4x_1 + 2x_2 = 6$

(vi) $\begin{cases} x_1 + x_2 = 5 \\ 2x_1 - x_2 = 4 \\ 3x_1 + 2x_2 = 13 \end{cases}$ (vii) $\begin{cases} x_1 + x_2 = 5 \\ 2x_1 - x_2 = 4 \\ 3x_1 + 2x_2 = 12 \end{cases}$

Solution. **(i)**

$$\xrightarrow{-3} \begin{cases} x_1 + 2x_2 = 4 \\ 3x_1 - x_2 = 5 \end{cases} \Leftrightarrow \xrightarrow{-\frac{1}{7}} \begin{cases} x_1 + 2x_2 = 4 \\ -7x_2 = -7 \end{cases}$$

$$\Leftrightarrow \xrightarrow{-2} \begin{cases} x_1 + 2x_2 = 4 \\ x_2 = 1 \end{cases} \Leftrightarrow \begin{cases} x_1 = 2 \\ x_2 = 1 \end{cases}$$

Answer: exactly one solution: $x_1 = 2$, $x_2 = 1$.

(Check: $2 + 2 \cdot 1 = 4$ and $3 \cdot 2 - 1 = 5$.)

(ii) To avoid fractions, we first produce an equation in which x_1 has the coefficient 1:

$$\xrightarrow{-2} \begin{cases} 2x_1 + 3x_2 = 1 \\ 5x_1 + 4x_2 = 13 \end{cases} \Leftrightarrow \begin{cases} 2x_1 + 3x_2 = 1 \\ x_1 - 2x_2 = 11 \end{cases} \Leftrightarrow \xrightarrow{-2} \begin{cases} x_1 - 2x_2 = 11 \\ 2x_1 + 3x_2 = 1 \end{cases}$$

$$\Leftrightarrow \xrightarrow{\frac{1}{7}} \begin{cases} x_1 - 2x_2 = 11 \\ 7x_2 = -21 \end{cases} \Leftrightarrow \xrightarrow{2} \begin{cases} x_1 - 2x_2 = 11 \\ x_2 = -3 \end{cases} \Leftrightarrow \begin{cases} x_1 = 5 \\ x_2 = -3 \end{cases}$$

Answer: exactly one solution: $x_1 = 5$, $x_2 = -3$.

(Check: $2 \cdot 5 + 3 \cdot (-3) = 1$ and $5 \cdot 5 + 4 \cdot (-3) = 13$.) Starting by multiplying the first equation with $\frac{1}{2}$ is equally fine; then fractions appear in between.

(iii)

$$\xrightarrow{\frac{1}{3}} \begin{cases} 3x_1 - 6x_2 = 9 \\ -2x_1 + 4x_2 = -6 \end{cases} \Leftrightarrow \xrightarrow{2} \begin{cases} x_1 - 2x_2 = 3 \\ -2x_1 + 4x_2 = -6 \end{cases} \Leftrightarrow \begin{cases} x_1 - 2x_2 = 3 \\ 0 = 0 \end{cases}$$

The equation $0 = 0$ can be omitted (the second equation was just -2 times the first one). In the remaining equation $x_1 - 2x_2 = 3$ the variable x_2 can be chosen arbitrarily: $x_2 = t$ with $t \in \mathbb{R}$, and then $x_1 = 3 + 2t$.

Answer: infinitely many solutions: $x_1 = 3 + 2t$, $x_2 = t$ for $t \in \mathbb{R}$.

(iv)

$$\xrightarrow{3} \begin{cases} 2x_1 - x_2 = 3 \\ -6x_1 + 3x_2 = 1 \end{cases} \Leftrightarrow \begin{cases} 2x_1 - x_2 = 3 \\ 0 = 10 \end{cases}$$

The equation $0 = 10$ is false for all x_1, x_2.

Answer: no solution.

(v) A linear system can also consist of a single equation.

$$\textcircled{\tfrac{1}{2}} \left\{ 4x_1 + 2x_2 = 6 \quad \Leftrightarrow \quad \right. \left\{ 2x_1 + x_2 = 3 \right.$$

Here x_1 can be chosen arbitrarily, $x_1 = t$ with $t \in \mathbb{R}$, and then $x_2 = 3 - 2t$.

Answer: infinitely many solutions: $x_1 = t$, $x_2 = 3 - 2t$ for $t \in \mathbb{R}$.

Equally correct: $x_1 = \frac{3}{2} - \frac{1}{2}s$, $x_2 = s$ for $s \in \mathbb{R}$ (choose x_2 as the free variable). This describes the same set of solutions (put $s = 3 - 2t$).

(vi)

$$\begin{array}{l} \textcircled{-3}\ \textcircled{-2} \\ \qquad\searrow\ \searrow \\ \qquad\qquad\searrow \end{array} \left\{ \begin{array}{rcl} x_1 + x_2 &=& 5 \\ 2x_1 - x_2 &=& 4 \\ 3x_1 + 2x_2 &=& 13 \end{array} \right. \Leftrightarrow \quad \textcircled{-\tfrac{1}{3}} \left\{ \begin{array}{rcl} x_1 + x_2 &=& 5 \\ -3x_2 &=& -6 \\ -x_2 &=& -2 \end{array} \right.$$

$$\Leftrightarrow \quad \begin{array}{l} \qquad\nearrow \\ \textcircled{1}\ \textcircled{-1} \\ \searrow \end{array} \left\{ \begin{array}{rcl} x_1 + x_2 &=& 5 \\ x_2 &=& 2 \\ -x_2 &=& -2 \end{array} \right. \Leftrightarrow \quad \left\{ \begin{array}{rcl} x_1 &=& 3 \\ x_2 &=& 2 \\ 0 &=& 0 \end{array} \right.$$

Answer: exactly one solution: $x_1 = 3$, $x_2 = 2$.

(Check: $3 + 2 = 5$, $6 - 2 = 4$, $9 + 4 = 13$.) Although there are more equations than variables, the system has a solution: the third equation holds automatically (it becomes $0 = 0$).

(vii) The same row operations as in (vi) give

$$\begin{array}{l} \textcircled{-3}\ \textcircled{-2} \\ \qquad\searrow\ \searrow \\ \qquad\qquad\searrow \end{array} \left\{ \begin{array}{rcl} x_1 + x_2 &=& 5 \\ 2x_1 - x_2 &=& 4 \\ 3x_1 + 2x_2 &=& 12 \end{array} \right. \Leftrightarrow \quad \textcircled{-\tfrac{1}{3}} \left\{ \begin{array}{rcl} x_1 + x_2 &=& 5 \\ -3x_2 &=& -6 \\ -x_2 &=& -3 \end{array} \right.$$

$$\Leftrightarrow \quad \begin{array}{l} \qquad\nearrow \\ \textcircled{1} \\ \searrow \end{array} \left\{ \begin{array}{rcl} x_1 + x_2 &=& 5 \\ x_2 &=& 2 \\ -x_2 &=& -3 \end{array} \right. \Leftrightarrow \quad \left\{ \begin{array}{rcl} x_1 + x_2 &=& 5 \\ x_2 &=& 2 \\ 0 &=& -1 \end{array} \right.$$

The equation $0 = -1$ is a contradiction. (The first two equations force $x_1 = 3$, $x_2 = 2$ as in (vi), but $3 \cdot 3 + 2 \cdot 2 = 13 \neq 12$.)

Answer: no solution.

Exercise 2. Find all solutions of the following linear systems in the variables x_1, x_2, x_3.

(i) $\left\{ \begin{array}{rcl} x_1 + 4x_2 - 2x_3 &=& 3 \\ x_2 + x_3 &=& 4 \\ 3x_3 &=& 6 \end{array} \right.$ (ii) $\left\{ \begin{array}{rcl} x_1 + x_2 + x_3 &=& 4 \\ 2x_1 + 3x_2 + x_3 &=& 4 \\ x_1 - x_2 + 2x_3 &=& 9 \end{array} \right.$

(iii) $\left\{ \begin{array}{rcl} x_1 - x_2 + 2x_3 &=& 1 \\ 2x_1 - x_2 + 5x_3 &=& 4 \end{array} \right.$ (iv) $\left\{ \begin{array}{rcl} x_1 + x_2 - x_3 &=& 1 \\ x_2 + 2x_3 &=& 3 \\ x_1 + 2x_2 + x_3 &=& 5 \end{array} \right.$

Solution. **(i)** Each equation contains one variable fewer than the one above it, so we can solve from the bottom to the top (**back substitution**): the third equation gives $x_3 = \frac{6}{3} = 2$; inserting this into the second equation gives $x_2 = 4 - x_3 = 2$; inserting both into the first equation gives $x_1 = 3 - 4x_2 + 2x_3 = 3 - 8 + 4 = -1$. (This is the same as eliminating x_3 and then x_2 from the equations above with (R1).)

Answer: exactly one solution: $x_1 = -1$, $x_2 = 2$, $x_3 = 2$.

(ii) First we eliminate x_1 from the second and third equation, then x_2 from the third equation:

$$\begin{matrix}(-1)\\(-2)\end{matrix}\begin{cases} x_1 + x_2 + x_3 = 4 \\ 2x_1 + 3x_2 + x_3 = 4 \\ x_1 - x_2 + 2x_3 = 9 \end{cases} \Leftrightarrow \begin{matrix}\\(2)\\\end{matrix}\begin{cases} x_1 + x_2 + x_3 = 4 \\ x_2 - x_3 = -4 \\ -2x_2 + x_3 = 5 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 + x_2 + x_3 = 4 \\ x_2 - x_3 = -4 \\ -x_3 = -3 \end{cases}$$

Back substitution: $x_3 = 3$, $x_2 = -4 + x_3 = -1$, $x_1 = 4 - x_2 - x_3 = 4 + 1 - 3 = 2$.

Answer: exactly one solution: $x_1 = 2$, $x_2 = -1$, $x_3 = 3$.

(Check: $2 - 1 + 3 = 4$, $4 - 3 + 3 = 4$, $2 + 1 + 6 = 9$.)

(iii)

$$\begin{matrix}(-2)\\\\\end{matrix}\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ 2x_1 - x_2 + 5x_3 = 4 \end{cases} \Leftrightarrow \begin{matrix}\\(1)\\\end{matrix}\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ x_2 + x_3 = 2 \end{cases} \Leftrightarrow \begin{cases} x_1 + 3x_3 = 3 \\ x_2 + x_3 = 2 \end{cases}$$

The variable x_3 can be chosen arbitrarily, $x_3 = t$ with $t \in \mathbb{R}$; then $x_1 = 3 - 3t$ and $x_2 = 2 - t$.

Answer: infinitely many solutions: $x_1 = 3 - 3t$, $x_2 = 2 - t$, $x_3 = t$ for $t \in \mathbb{R}$.

(iv)

$$\begin{matrix}(-1)\\\\\end{matrix}\begin{cases} x_1 + x_2 - x_3 = 1 \\ x_2 + 2x_3 = 3 \\ x_1 + 2x_2 + x_3 = 5 \end{cases} \Leftrightarrow \begin{matrix}\\(-1)\\\end{matrix}\begin{cases} x_1 + x_2 - x_3 = 1 \\ x_2 + 2x_3 = 3 \\ x_2 + 2x_3 = 4 \end{cases} \Leftrightarrow \begin{cases} x_1 + x_2 - x_3 = 1 \\ x_2 + 2x_3 = 3 \\ 0 = 1 \end{cases}$$

The equation $0 = 1$ is a contradiction (the expression $x_2 + 2x_3$ cannot be equal to 3 and to 4 at the same time).

Answer: no solution.

Exercise 3. At the Nagoya University festival, a student club sells takoyaki and okonomiyaki. One portion of takoyaki needs 100 g of flour, 1 egg and 100 g of cabbage; one okonomiyaki needs 200 g of flour, 1 egg and 300 g of cabbage.

(a) On the first day, the club used 15 kg of flour, 105 eggs and 19.5 kg of cabbage. Describe this by a linear system and find out how many portions of takoyaki and how many okonomiyaki were sold.

(b) One portion of takoyaki costs 500 ¥ and one okonomiyaki 700 ¥. How much money did the club earn on the first day?

(c) For the second day, the club wrote down 16 kg of flour, 110 eggs and 19 kg of cabbage. Show that someone must have made a mistake when writing down these amounts.

Solution. **(a)** Let x_1 be the number of portions of takoyaki and x_2 the number of okonomiyaki. Counting the flour (in g), the eggs and the cabbage (in g) gives three linear equations:

$$\begin{matrix}(\frac{1}{100})\\\\(\frac{1}{100})\end{matrix}\begin{cases} 100x_1 + 200x_2 = 15000 \\ x_1 + x_2 = 105 \\ 100x_1 + 300x_2 = 19500 \end{cases} \Leftrightarrow \begin{matrix}(-1)\\\\\\\end{matrix}\begin{matrix}(-1)\\\\\\\end{matrix}\begin{cases} x_1 + 2x_2 = 150 \\ x_1 + x_2 = 105 \\ x_1 + 3x_2 = 195 \end{cases}$$

$$\Leftrightarrow \begin{matrix}\\(1)\\\end{matrix}\begin{cases} x_1 + 2x_2 = 150 \\ -x_2 = -45 \\ x_2 = 45 \end{cases} \Leftrightarrow \begin{matrix}\\(-1)\\\end{matrix}\begin{cases} x_1 + 2x_2 = 150 \\ -x_2 = -45 \\ 0 = 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{matrix}\\ \scriptstyle -2\end{matrix}\!\!\begin{cases} x_1 + 2x_2 = 150 \\ x_2 = 45 \end{cases} \Leftrightarrow \quad \begin{cases} x_1 = 60 \\ x_2 = 45 \end{cases}$$

(In the second-to-last step the equation $0 = 0$ was omitted.)

Answer: $x_1 = 60$ portions of takoyaki and $x_2 = 45$ okonomiyaki were sold.

(Check: $6000 + 9000 = 15000$ g of flour, $60 + 45 = 105$ eggs, $6000 + 13500 = 19500$ g of cabbage.)

(b) $60 \cdot 500 + 45 \cdot 700 = 30000 + 31500 = 61500$.

Answer: The club earned 61 500 ¥.

(c) The same row operations as in (a) give

$$\begin{matrix}\scriptstyle \frac{1}{100}\\ \\ \scriptstyle \frac{1}{100}\end{matrix}\!\!\begin{cases} 100x_1 + 200x_2 = 16000 \\ x_1 + x_2 = 110 \\ 100x_1 + 300x_2 = 19000 \end{cases} \Leftrightarrow \quad \begin{matrix}\scriptstyle -1\\ \\ \end{matrix}\!\!\begin{matrix}\scriptstyle -1\\ \\ \end{matrix}\!\!\begin{cases} x_1 + 2x_2 = 160 \\ x_1 + x_2 = 110 \\ x_1 + 3x_2 = 190 \end{cases}$$

$$\Leftrightarrow \quad \begin{matrix}\scriptstyle 1\\ \\ \end{matrix}\!\!\begin{cases} x_1 + 2x_2 = 160 \\ -x_2 = -50 \\ x_2 = 30 \end{cases} \Leftrightarrow \quad \begin{cases} x_1 + 2x_2 = 160 \\ -x_2 = -50 \\ 0 = -20 \end{cases}$$

The equation $0 = -20$ is a contradiction. (Flour and eggs give $x_2 = 50$ and $x_1 = 60$, but then $6 + 15 = 21$ kg of cabbage would have been used, not 19 kg.)

Answer: the linear system has no solution, so these amounts cannot all be correct.

Exercise 4.

(a) Give a linear system of two equations in the variables x_1, x_2 which has

1. exactly one solution, namely $x_1 = 2$, $x_2 = -1$;
2. no solution;
3. infinitely many solutions, one of them being $x_1 = 2$, $x_2 = -1$.

(b) True or false? Give a reason or a counterexample.

1. A linear system with more equations than variables has no solution.
2. A linear system with fewer equations than variables has infinitely many solutions.
3. A linear system in which all right-hand sides are 0 has at least one solution.

Solution. **(a)** There are many possible answers, for example the following. (How to find them: choose the left-hand sides, e.g. $x_1 + x_2$ and $x_1 - x_2$, and compute the right-hand sides by inserting $x_1 = 2$, $x_2 = -1$.)

1. Exactly one solution:

$$\begin{matrix}\scriptstyle -1\\ \\ \end{matrix}\!\!\begin{cases} x_1 + x_2 = 1 \\ x_1 - x_2 = 3 \end{cases} \Leftrightarrow \quad \begin{matrix}\\ \scriptstyle -\frac{1}{2}\end{matrix}\!\!\begin{cases} x_1 + x_2 = 1 \\ -2x_2 = 2 \end{cases}$$

$$\Leftrightarrow \quad \begin{matrix}\\ \scriptstyle -1\end{matrix}\!\!\begin{cases} x_1 + x_2 = 1 \\ x_2 = -1 \end{cases} \Leftrightarrow \quad \begin{cases} x_1 = 2 \\ x_2 = -1 \end{cases}$$

2. No solution: the same left-hand side with two different right-hand sides,

$$\begin{matrix}\scriptstyle -1\\ \\ \end{matrix}\!\!\begin{cases} x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases} \Leftrightarrow \quad \begin{cases} x_1 + x_2 = 1 \\ 0 = 1 \end{cases}$$

3. Infinitely many solutions: the second equation is a multiple of the first one,

$$\begin{array}{l} \textcircled{-2} \\ \hookrightarrow \end{array} \begin{cases} x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 2 \end{cases} \Leftrightarrow \quad \begin{cases} x_1 + x_2 = 1 \\ 0 = 0 \end{cases}$$

All solutions: $x_1 = 1 - t$, $x_2 = t$ for $t \in \mathbb{R}$; for $t = -1$ we get $x_1 = 2$, $x_2 = -1$.

(b) 1. False. The system in Exercise 1 (vi) has three equations, two variables and exactly one solution.

2. False. For example, $\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 1 \end{cases}$ has two equations, three variables and no solution (the expression $x_1 + x_2 + x_3$ cannot be 0 and 1 at the same time).

3. True. $x_1 = 0, \ldots, x_n = 0$ is always a solution, since $a_1 \cdot 0 + \cdots + a_n \cdot 0 = 0$ for every equation.