Quiz 3: Rank & functions – Solutions

Tutorial 3, Tuesday, 20th October, 2026

Exercise 1. (4 Points) Calculate the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 5 & 1 \\ 3 & 7 & 0 \end{pmatrix}.$$

Show your working.

Solution. We use row operations to obtain the row-reduced echelon form:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 5 & 1 \\ 3 & 7 & 0 \end{pmatrix} \overset{\textcircled{-3}\ \textcircled{-2}}{\sim} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{pmatrix} \overset{\textcircled{-1}}{\sim} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \overset{\textcircled{-2}}{\sim} \begin{pmatrix} 1 & 0 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{rref}(A).$$

There are two pivot elements (equivalently, two nonzero rows) in the row-reduced echelon form. Hence

$$\boxed{\operatorname{rk}(A) = 2.}$$

Exercise 2. (3+3=6 Points) For each of the following functions, decide whether it is injective and whether it is surjective. Justify your answers.

(i)
$$f\colon \mathbb{R} \longrightarrow \mathbb{R}, \qquad x \longmapsto e^{2x} - 1\,.$$

(ii)
$$g\colon \mathbb{R}^2 \longrightarrow \mathbb{R}, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto x_1 - 3x_2\,.$$

Solution.

(i) If $f(x) = f(x')$, then $e^{2x} = e^{2x'}$. Taking logarithms gives $2x = 2x'$, so $x = x'$. Thus f is injective.

Since $e^{2x} > 0$ for all $x \in \mathbb{R}$, we have $f(x) > -1$. In particular, $-1 \in \mathbb{R}$ is not attained, so f is not surjective.

$$\boxed{f \text{ is injective but not surjective.}}$$

(ii) Let $y \in \mathbb{R}$ be arbitrary. Then

$$g\left(\begin{pmatrix} y \\ 0 \end{pmatrix}\right) = y - 3 \cdot 0 = y.$$

Thus every element of the codomain has a preimage, so g is surjective. However,

$$g\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = 0 = g\left(\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right), \qquad \begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Hence g is not injective.

$$\boxed{g \text{ is surjective but not injective.}}$$