Linear Algebra I & Mathematics Tutorial 1b — Instructor: Henrik Bachmann
Nagoya University, G30 Program, Fall 2024 — TAs: Panagiotis Jones, Xianghang Zhang

Tutorial 9: Inverses and subspaces

Exercise 1. Decide if the following two linear maps are invertible. Determine their inverses if they exist.

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_2 + 2x_3 \\ x_1 - 4x_2 + 6x_3 \\ x_2 + x_3 \end{pmatrix}, \qquad G : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 10x_1 + x_2 - 26x_3 \\ x_1 - 2x_3 \\ -x_1 + x_3 \end{pmatrix}.$$

Exercise 2. (Final Exam 2021) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 1 & 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & -2 \\ -1 & 2 & 5 \end{pmatrix}$.

$A^{-1} = \begin{pmatrix} \frac{9}{2} & -2 & -\frac{7}{2} \\ -1 & 1 & 1 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$

(i) Determine whether or not the matrices A and B are invertible and, if they are, compute their inverses.

B not invertible

(ii) Calculate the matrix BA and decide if BA^n is invertible for any integer $n \geq 1$.

A invertible $\Rightarrow A^n$ inv.
If BA^n inv. then $B = BA^n \cdot (A^n)^{-1}$ would be inv.
$\Rightarrow BA^n$ not inv.

A subset $U \subset \mathbb{R}^n$ is a **subspace** of $\mathbb{R}^n$ if

i) $0 \in U$,

ii) for all $u, v \in U$ we have $u + v \in U$,

iii) for all $u \in U$ and $\lambda \in \mathbb{R}$ we have $\lambda u \in U$.

The **span** of $v_1, \dots, v_n \in \mathbb{R}^m$ is the set

$$\operatorname{span}\{v_1, \dots, v_n\} = \{\lambda_1 v_1 + \dots + \lambda_n v_n \in \mathbb{R}^m \mid \lambda_1, \dots, \lambda_n \in \mathbb{R}\}.$$

Exercise 3. (Final Exam 2019) Which of the following subsets of $\mathbb{R}^2$ are subspaces? Justify your answers.

(i) $U_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 + x_2 = x_1 x_2 \right\}$. No

(ii) $U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid 2x_1 = x_1 + x_2 \right\}$. Yes

(iii) $U_3 = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \cup \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$. No

(Reminder: $\cup$ is the union of two sets)

(Solutions for Exercise 2 & 3 are contained in the solutions for the Final exam 2021 & 2019)

Homework 4: Reflection, Projection and Inverses

Deadline: 10th December, 2024

Exercise 1. (2+3 = 5 Points) Let $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$ be with $u \neq 0$.

(i) Show that the reflection $\rho_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear map.

(ii) Show that the matrix of the projection $P_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is given by

$$[P_u] = \frac{1}{u \bullet u} u u^T \in \mathbb{R}^{n \times n},$$

where $u^T = (u_1\, u_2\, \ldots\, u_n) \in \mathbb{R}^{1 \times n}$. Use this to give an expression for $[\rho_u]$.

Exercise 2. (2+3 = 5 Points) Let $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $d = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $x = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

(i) Calculate the matrices $[P_u]$ and $[\rho_u]$ in this special case.

(ii) Calculate the following vectors and draw them in one picture together with u,d and x

$$P_u(x), \quad \rho_u(x), \quad (P_u \circ P_d)(x), \quad \mathrm{rot}_{\frac{\pi}{2}}(x), \quad (P_u \circ \mathrm{rot}_{\frac{\pi}{2}})(x), \quad (\mathrm{rot}_{\frac{\pi}{2}} \circ P_u)(x).$$

Exercise 3. (2+2 = 4 Points) Show that for all $u \in \mathbb{R}^n$ with $u \neq 0$ the projection P_u and the reflection ρ_u satisfy for all $x \in \mathbb{R}^n$ the following two properties:

(i) $P_u(P_u(x)) = P_u(x)$.

(ii) $\rho_u(\rho_u(x)) = x$.

Exercise 4. (3+3= 6 Points)

(i) Decide if the following two linear maps are invertible. Determine their inverses if they exist.

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_2 + 2x_3 \\ x_1 - 4x_2 + 6x_3 \\ x_2 + x_3 \end{pmatrix},$$

$$G : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 4x_1 + x_2 - 2x_3 \\ x_1 - 2x_2 - 4x_3 \\ -x_1 + x_3 \end{pmatrix}.$$

(ii) The **kernel** of a linear map $H : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$\ker(H) = \{x \in \mathbb{R}^n \mid H(x) = 0\}.$$

Determine $\ker(F)$ and $\ker(G)$.

Exercise 1. Decide if the following two linear maps are invertible. Determine their inverses if they exist.

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_2 + 2x_3 \\ x_1 - 4x_2 + 6x_3 \\ x_2 + x_3 \end{pmatrix}, \qquad G : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 10x_1 + x_2 - 26x_3 \\ x_1 - 2x_3 \\ -x_1 + x_3 \end{pmatrix}.$$

The matrix of F is $[F] = \begin{pmatrix} 0 & 2 & 2 \\ 1 & -4 & 6 \\ 0 & 1 & 1 \end{pmatrix}$. We calculate:

$$([F] \mid I_3) = \left(\begin{array}{ccc|ccc} 0 & 2 & 2 & 1 & 0 & 0 \\ 1 & -4 & 6 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -4 & 6 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -4 & 6 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 0 \\ \underbrace{0 \quad 0 \quad 0} & & & -\frac{1}{2} & 0 & 1 \end{array}\right)$$

Here we already see that $\mathrm{rref}([F]) \neq I_3$ and therefore F is not invertible.

For G we have $[G] = \begin{pmatrix} 10 & 1 & -26 \\ 1 & 0 & -2 \\ -1 & 0 & 1 \end{pmatrix}$ and we get

$$([G] \mid I_3) = \left(\begin{array}{ccc|ccc} 10 & 1 & -26 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 1 & 0 \\ 10 & 1 & -26 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & -6 & 1 & -10 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 0 & -1 & -2 \\ 1 & -16 & -6 \\ 0 & -1 & -1 \end{matrix}}_{[G]^{-1}} \\ 0 & 0 & 1 & \end{array}\right)$$

Therefore G is invertible and the inverse is given by

$$G^{-1}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} -x_2 - 2x_3 \\ x_1 - 16x_2 - 6x_3 \\ -x_2 - x_3 \end{pmatrix} = [G]^{-1} x .$$

Additional question: What is $\ker(F)$, $\ker(G)$?

$$\ker(F) = \{ x \in \mathbb{R}^3 \mid F(x) = 0 \}$$

By above calculation:

$$[F] \sim \frac{1}{2} \begin{pmatrix} 1 & -4 & 6 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \overset{\curvearrowright}{4} \begin{pmatrix} 1 & -4 & 6 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 10 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Solutions to $F(x) = 0$:

$$x_1 = -10t$$
$$x_2 = -t$$
$$x_3 = t$$

$$\ker(F) = \left\{ x \in \mathbb{R}^3 \;\middle|\; x = t \begin{pmatrix} -10 \\ -1 \\ 1 \end{pmatrix}, t \in \mathbb{R} \right\}$$

Since G is invertible, $G(x)=0$ just has the solution $x=0$, i.e.

$$\ker(G) = \{0\}.$$