Tutorial 9: Inverses and subspaces

Exercise 1. Decide if the following two linear maps are invertible. Determine their inverses if they exist.

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 , \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_2 + 2x_3 \\ x_1 - 4x_2 + 6x_3 \\ x_2 + x_3 \end{pmatrix} ,$$

$$G : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 , \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 10x_1 + x_2 - 26x_3 \\ x_1 - 2x_3 \\ -x_1 + x_3 \end{pmatrix} .$$

Exercise 2. (Final Exam 2021) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 1 & 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & -2 \\ -1 & 2 & 5 \end{pmatrix}$.

(i) Determine whether or not the matrices A and B are invertible and, if they are, compute their inverses.

(ii) Calculate the matrix BA and decide if BA^n is invertible for any integer $n \geq 1$.

A subset $U \subset \mathbb{R}^n$ is a **subspace** of $\mathbb{R}^n$ if

i) $0 \in U$,

ii) for all $u, v \in U$ we have $u + v \in U$,

iii) for all $u \in U$ and $\lambda \in \mathbb{R}$ we have $\lambda u \in U$.

The **span** of $v_1, \dots, v_n \in \mathbb{R}^m$ is the set

$$\operatorname{span}\{v_1, \dots, v_n\} = \{\lambda_1 v_1 + \cdots + \lambda_n v_n \in \mathbb{R}^m \mid \lambda_1, \dots, \lambda_n \in \mathbb{R}\} .$$

Exercise 3. (Final Exam 2019) Which of the following subsets of $\mathbb{R}^2$ are subspaces? Justify your answers.

(i) $U_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 + x_2 = x_1 x_2 \right\}$.

(ii) $U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid 2x_1 = x_1 + x_2 \right\}$.

(iii) $U_3 = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \bigcup \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

(Reminder: $\cup$ is the union of two sets)

(Solutions for Exercise 2 & 3 are contained in the solutions for the Final exam 2021 & 2019)