Homework 4: Reflection, Projection and Inverses

Deadline: 10th December, 2024

Exercise 1. (2+3 = 5 Points) Let $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$ be with $u \neq 0$.

(i) Show that the reflection $\rho_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear map.

(ii) Show that the matrix of the projection $P_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is given by

$$[P_u] = \frac{1}{u \bullet u} u u^T \in \mathbb{R}^{n \times n},$$

where $u^T = (u_1\, u_2\, \dots\, u_n) \in \mathbb{R}^{1 \times n}$. Use this to give an expression for $[\rho_u]$.

Exercise 2. (2+3 = 5 Points) Let $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $d = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $x = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

(i) Calculate the matrices $[P_u]$ and $[\rho_u]$ in this special case.

(ii) Calculate the following vectors and draw them in one picture together with u,d and x

$$P_u(x), \quad \rho_u(x), \quad (P_u \circ P_d)(x), \quad \mathrm{rot}_{\frac{\pi}{2}}(x), \quad (P_u \circ \mathrm{rot}_{\frac{\pi}{2}})(x), \quad (\mathrm{rot}_{\frac{\pi}{2}} \circ P_u)(x).$$

Exercise 3. (2+2 = 4 Points) Show that for all $u \in \mathbb{R}^n$ with $u \neq 0$ the projection P_u and the reflection ρ_u satisfy for all $x \in \mathbb{R}^n$ the following two properties:

(i) $P_u(P_u(x)) = P_u(x)$.

(ii) $\rho_u(\rho_u(x)) = x$.

Exercise 4. (3+3= 6 Points)

(i) Decide if the following two linear maps are invertible. Determine their inverses if they exist.

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_2 + 2x_3 \\ x_1 - 4x_2 + 6x_3 \\ x_2 + x_3 \end{pmatrix},$$

$$G : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 4x_1 + x_2 - 2x_3 \\ x_1 - 2x_2 - 4x_3 \\ -x_1 + x_3 \end{pmatrix}.$$

(ii) The **kernel** of a linear map $H : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$\ker(H) = \{x \in \mathbb{R}^n \mid H(x) = 0\}.$$

Determine $\ker(F)$ and $\ker(G)$.

1) (10 Points) Consider the following linear system

$$\left\{\begin{array}{rcrcrcrcr} -x_1 & + & 2x_2 & + & 3x_3 & + & 5x_4 & = & 2 \\ 2x_1 & + & x_2 & & & + & x_4 & = & 1 \\ 3x_1 & - & 2x_2 & + & 2x_3 & & & = & -2 \end{array}\right. .$$

(i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

(ii) Calculate the row-reduced echelon forms of the matrices $(A \mid b)$ and A, and determine their ranks.

(iii) Determine all the solutions to the linear system $Ax = b$.

(iv) Find all vectors $x \in \mathbb{R}^4$ with $\|x\| = 1$ and $Ax = b$.

2) (8 Points) Let $u = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \in \mathbb{R}^3$ and define the following three functions:

$$f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad x \longmapsto (x \bullet u)u + x\,, \qquad f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto e^{x_1}\sin(x_2)\,, \qquad f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad x \longmapsto \begin{pmatrix} 0 \\ \|x\| \end{pmatrix}.$$

(i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.

(ii) Is f_2 injective and/or surjective?

3) (8 Points) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad G\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

(i) Determine the matrix of G.

(ii) Determine the matrix of $G \circ G \circ G \circ G \circ G \circ G \circ G \circ G \circ G \circ G$.

(iii) Find all vectors $x \in \mathbb{R}^2$ such that $G(x) = -x$.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$$
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 + x_3 \\ x_1 - x_2 - x_3 \\ x_1 \\ x_2 + x_3 \end{pmatrix}.$$

(i) Calculate the image of H.

(ii) Decide if H is injective and/or surjective.

(iii) We define the kernel of H as the following set

$$\ker(H) = \{x \in \mathbb{R}^3 \mid H(x) = 0\}.$$

Determine $\ker(H)$.

(iv) Find all vectors $x \in \mathbb{R}^3$ such that $x \bullet v = 0$ for all $v \in \ker(H)$.

Homework 4:

Exercise 1. (2+3 = 5 Points) Let $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$ be with $u \neq 0$.

(i) Show that the reflection $\rho_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear map.

(ii) Show that the matrix of the projection $P_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is given by

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where $u^T = (u_1\, u_2\, \ldots\, u_n) \in \mathbb{R}^{1 \times n}$. Use this to give an expression for $[\rho_u]$.

(i)
$$\rho_u(x) = 2 \frac{u \bullet x}{u \bullet u} u - x$$

Check $\rho_u(x+y) = \rho_u(x) + \rho_u(y)$
and $\rho_u(\lambda x) = \lambda \rho_u(x)$ by using properties of "$\bullet$"

(ii)
- Show that the i-th column of $\frac{1}{u \bullet u} u u^T$ is given be $P_u(e_i)$.
- Show $[\rho_u] = 2[P_u] - I_n$.

Exercise 2 & 3: direct calculation.

Exercise 4. (3+3= 6 Points)

(i) Decide if the following two linear maps are invertible. Determine their inverses if they exist.

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_2 + 2x_3 \\ x_1 - 4x_2 + 6x_3 \\ x_2 + x_3 \end{pmatrix}, \qquad G : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 4x_1 + x_2 - 2x_3 \\ x_1 - 2x_2 - 4x_3 \\ -x_1 + x_3 \end{pmatrix}.$$

(ii) The **kernel** of a linear map $H : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$\ker(H) = \{x \in \mathbb{R}^n \mid H(x) = 0\}.$$

Determine $\ker(F)$ and $\ker(G)$.

Will be explained in Lecture

Spoiler: To check if $F: \mathbb{R}^3 \to \mathbb{R}^3$ is invertible we want to consider $([F] \mid I_n)$ and bring it onto rref. If we end up with $(I_n \mid B)$ then $[F^{-1}] = B$.

Example for Exercise 4:

Determine (if it exists) the inverse of $F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ 2x_1 - x_2 \end{pmatrix}$$

Want to solve $F(x) = y$ for $\underbrace{\text{any}}_{}$ y and show that the solution is unique. Then $F^{-1}(y) = x$.

F surjective

F injective

$$\left(\begin{array}{cc|c} 1 & 1 & y_1 \\ 2 & -1 & y_2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & y_1 \\ 0 & -3 & -2y_1+y_2 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 1 & y_1 \\ 0 & 1 & \frac{2}{3}y_1+\frac{1}{3}y_2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{3}y_1-\frac{1}{3}y_2 \\ 0 & 1 & \frac{2}{3}y_1+\frac{1}{3}y_2 \end{array}\right).$$

The map $G: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longmapsto \begin{pmatrix} \frac{1}{3}y_1 - \frac{1}{3}y_2 \\ \frac{2}{3}y_1 + \frac{1}{3}y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

satisfies $F(G(y)) = y$ for any $y \in \mathbb{R}^2$.

$G = F^{-1}$ is the inverse of F.