

## Tutorial 6: Review for the midterm exam

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### General comments:

- (i) This Friday, we have Lecture 7. Please make sure to watch the videos for Lecture 6 beforehand (see last week's TACT announcement).
- (ii) Next Friday, November 22nd, we will have the midterm exam, covering content from Lectures 1-7.
- (iii) You can find the midterm exams and solutions from the past four years on the homepage.
- (iv) For Homework 3, Exercise 1, the videos "Image of a Non-Linear Map" and "When is a Linear Map Surjective/Injective" might be helpful. (These can be found under "Additional Materials" on the homepage.)

### Exercise 1. Consider the linear map

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$x \longmapsto \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 2 \\ 1 & 1 & -5 \end{pmatrix} x.$$

- (i) Calculate  $\text{im}(F)$ .
- (ii) Is  $F$  surjective and/or injective?
- (iii) Find all solutions to  $F(x) = 0$ .
- (iv) Find all  $x \in \mathbb{R}^3$  such that  $v \bullet x = 0$  for all  $v \in \text{im}(F)$ .

Recall: In (iv) the **dot product**  $\bullet$  of  $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$  is defined by

$$u \bullet v = u_1 v_1 + \cdots + u_n v_n.$$

The solutions for Exercises 2-7 below can be found in "Tutorial 6 2019" on the homepage.

### Exercise 2. Give an example of a linear system which has

- (i) exactly one solution.
- (ii) infinitely many solutions.
- (iii) no solutions.

### Exercise 3. Which of the following matrices are on row-reduced echelon form?

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (1 \quad 2 \quad 3), \quad \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

**Exercise 4.** Consider the following linear system

$$\begin{cases} x_1 + 4x_2 + 7x_3 + 2x_4 = 1 \\ 2x_1 + 5x_2 + 8x_3 + x_4 = 2 \\ 3x_1 + 6x_2 + 10x_3 + x_4 = 1 \end{cases} .$$

- (i) Find a matrix  $A \in \mathbb{R}^{3 \times 4}$  and a vector  $b \in \mathbb{R}^3$ , such that the solutions of the above linear system are given by the vectors  $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$  satisfying  $Ax = b$ .
- (ii) Calculate the row-reduced echelon form of  $(A | b)$ .
- (iii) Find all the solutions to the linear system.
- (iv) Calculate the rank of  $(A | b)$  and  $A$ .

**Exercise 5.** Define for a matrix  $A \in \mathbb{R}^{m \times n}$  the linear map

$$\begin{aligned} F_A : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ x &\longmapsto Ax . \end{aligned}$$

Choose for  $A$  the matrices appearing in Exercise 2 & 3 and decide for each case if  $F_A$  is injective and/or surjective and calculate the image of  $F_A$ .

**Exercise 6.**

- (i) Determine all vectors that are orthogonal to the vector  $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ .
- (ii) Find a linear map  $G : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ , such that the image of  $G$  is given by all the vectors which are orthogonal to  $v$ . (i.e. the image gives all the vectors you determined in i) ).

**Exercise 7.** Let  $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$  and define the following four functions:

$$\begin{aligned} f_1 : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \text{rot}_x(u) , \end{aligned} \qquad \begin{aligned} f_2 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \begin{pmatrix} u \bullet x \\ (u \bullet u)(x \bullet u) \end{pmatrix} , \end{aligned}$$

$$\begin{aligned} f_3 : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &\longmapsto \begin{pmatrix} x_1 + x_2 + x_3 \\ 1 \end{pmatrix} , \end{aligned} \qquad \begin{aligned} f_4 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\longmapsto \begin{pmatrix} 2x_2 - x_1 \\ 2x_1 \\ 4x_1 + x_2 \end{pmatrix} . \end{aligned}$$

- (i) Give a geometric interpretation of the image of  $f_1$ . Is  $f_1$  injective and/or surjective?
- (ii) Which of the above functions  $f_1, f_2, f_3, f_4$  are linear maps? For each one that is linear, determine its matrix.