Tutorial 15: Orthogonal complement & Normal equation

Exercise 1. We define the subspace $U = \text{span}\{u_1, u_2, u_3\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

(i) Determine a basis $B = (b_1, \ldots, b_m)$ of U and calculate its dimension.

(ii) Determine a basis for $U^\perp$.

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix $A = [F] \in \mathbb{R}^{m \times n}$ and let $y \in \mathbb{R}^m$ be an arbitrary vector.

If $y \in \text{im}(F)$ then the linear system $Ax = y$ has a solution. But if $y \notin \text{im}(F)$ then there does not exists a $x \in \mathbb{R}^n$ with $Ax = y$. In this case, we can ask for the best possible x, i.e. the one such that $\|Ax - y\|$ is minimal.

Facts:

(i) The $x \in \mathbb{R}^n$ such that $\|Ax - y\|$ is minimal is given by a solution of the **normal equation**

$$A^T Ax = A^T y.$$

(ii) The normal equation always has (at least one) solution x. This x has the property $Ax = P_{\text{im}(F)}(y)$, i.e. Ax is the orthogonal projection of y onto the image of F.

(iii) If $\ker(A) = \{0\}$ (the columns of A are linearly independent) then $A^T A \in \mathbb{R}^{n \times n}$ is invertible and the normal equation has a unique solution given by

$$x = (A^T A)^{-1} A^T y.$$

(But in real-life, e.g. your final exam, you would rather just directly solve $A^T Ax = A^T y$ in the classical way rather than calculating $(A^T A)^{-1}$)

Exercise 2. Assume we have the following data points

i	1	2	3
x_i	0	1	2
y_i	2	1	3

Find the line of best fit for the above data, i.e. find $a, b \in \mathbb{R}$ such that the function $l(x) = ax + b$ minimizes the sum of squares $\sum_{i=1}^{3} (l(x_i) - y_i)^2$.