Homework 6: Linear independence & Basis

Deadline: 14th January, 2025

Exercise 1. (6 Points) Let $V \subset \mathbb{R}^n$ be a subspace, $v_1, \dots, v_l \in V$ linearly independent and $V = \operatorname{span}\{w_1, \dots, w_m\}$ for some $w_1, \dots, w_m \in \mathbb{R}^n$. Show that we have $l \leq m$. (Without using Lemma 9.4)

In other words: Show that a subspace spanned by m vectors can not contain more than m linearly independent vectors.

Exercise 2. (7 Points) Determine bases for the kernel and the image of the following linear map

$$\begin{aligned} F : \mathbb{R}^5 &\longrightarrow \mathbb{R}^3 \\ x &\longmapsto \begin{pmatrix} 2 & 4 & 1 & -9 & 2 \\ 3 & 6 & 3 & -6 & 3 \\ 1 & 2 & 2 & 3 & 1 \end{pmatrix} x \,. \end{aligned}$$

The following exercise is intended to show the basic idea of 3D computer graphics, by showing how to get a 2-dimensional picture (to be shown on a 2-dimensional monitor) from an 3-dimensional object.

Exercise 3. (7 Points)

(i) We define the corners of a cube with side length 18 in $\mathbb{R}^3$ by the following set of 8 points:

$$W = \left\{ \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3 \mid w_1, w_2, w_3 \in \{0, 18\} \right\}.$$

Make a drawing of a cube with side length 18 in $\mathbb{R}^3$, i.e. draw the 8 points in the set W and connect two points if they differ just by one entry.

(This just means that you draw a cube like you would usually draw it. "Differ by one entry" just means that these points are on the same edge of the cube.)

(ii) Show that $D = (d_1, d_2, d_3)$ is a basis of $\mathbb{R}^3$, where

$$d_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad d_2 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \quad d_3 = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix}.$$

(iii) Write each $x \in W$ as a linear combination in the basis D, i.e. for each x find $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ with

$$x = \lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_3 \,.$$

(iv) For each $x \in W$ draw the points (λ_1, λ_2) in $\mathbb{R}^2$. Connect two points if the corresponding elements in W just differ by one entry.

Explanation: What you should get in (iv) is a drawing of the 3-dimensional cube in 2 dimensions. The basis D somehow describes from which direction you look at the cube. If you replaced the D by the standard basis (e_1, e_2, e_3), you would get a picture of the cube from the top (i.e., just a square). The λ_3, which you did not use for the drawing, describes the distance in the viewing direction.

Exercise 1. (6 Points) Let $V \subset \mathbb{R}^n$ be a subspace, $v_1, \dots, v_l \in V$ linearly independent and $V = \operatorname{span}\{w_1, \dots, w_m\}$ for some $w_1, \dots, w_m \in \mathbb{R}^n$. Show that we have $l \leq m$. (Without using Lemma 9.4)

In other words: Show that a subspace spanned by m vectors can not contain more than m linearly independent vectors.

Hint: Show that $l > m$ implies that $v_1, \dots, v_l$ are lin. dependent

$\iff \begin{pmatrix} | & & | \\ v_1 & \cdots & v_l \\ | & & | \end{pmatrix} x = 0$ has infinitely many solutions.

Exercise 2. (7 Points) Determine bases for the kernel and the image of the following linear map

$$
\begin{aligned}
F : \mathbb{R}^5 &\longrightarrow \mathbb{R}^3 \\
x &\longmapsto \left(\begin{pmatrix} 2 & 4 & 1 & -9 & 2 \\ 3 & 6 & 3 & -6 & 3 \\ 1 & 2 & 2 & 3 & 1 \end{pmatrix} \right) x \,.
\end{aligned}
$$

$$
[F] \sim \dots \sim \begin{array}{ccccc} & F & F & & F \\ \hline \end{array}
\begin{pmatrix} 1 & 2 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

wrong and just example

Pivot positions: 1, 4 th column $\Rightarrow \left\{ \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -9 \\ -6 \\ 3 \end{pmatrix} \right\}$ basis of $\operatorname{im}(F)$

$x \in \ker(F) \iff x = t_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} -4 \\ 0 \\ 0 \\ -5 \\ 1 \end{pmatrix}$

3 free variables

Basis of $\operatorname{Ker}(F)$