1) (10 Points) Consider the following linear system

$$\begin{cases} -2x_1 + 4x_2 + x_3 + x_4 = 6 \\ -3x_1 + 6x_2 + x_3 = 7 \\ x_1 - 2x_2 + x_4 = -1 \end{cases}.$$

i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

ii) Determine the row-reduced echelon forms of the matrices $(A \mid b)$ and A.

iii) Find all the solutions to the linear system.

iv) Calculate the rank of $(A \mid b)$ and A.

v) Find all $y \in \mathbb{R}^4$ with $Ay = 2b$ by using your result for iii).

2) (8 Points) Let $u = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \in \mathbb{R}^2$ and define the following four functions:

$$f_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \quad x \longmapsto \begin{pmatrix} u \bullet x \\ 0 \\ x \bullet u \end{pmatrix}, \qquad f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto 2^{x_1+x_2} - 1, \qquad f_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 - 3x_2 \\ 2x_1 + x_2x_3 \end{pmatrix}.$$

i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.

ii) Is f_2 injective and/or surjective?

3) (8 Points)

i) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad G\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Determine the matrix of G.

ii) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a function with

$$F\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad F\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad F\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Show that F is <u>not</u> a linear map.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_1 - x_3 \\ x_2 + x_3 \end{pmatrix}.$$

i) Calculate the image of H.

ii) Decide if H is injective and/or surjective.

iii) Find a non-zero vector $v \in \mathbb{R}$, such that v is orthogonal to $H(v)$. (Just <u>one explicit</u> vector is enough)

After finishing this exam please send your solution as one pdf file to
henrik.bachmann@math.nagoya-u.ac.jp

1) (10 Points) Consider the following linear system

$$\begin{cases} 3x_1 - 6x_2 + x_3 + 5x_4 = 5 \\ 2x_1 - 4x_2 + x_3 + 3x_4 = 4 \\ -x_1 + 2x_2 - 2x_3 = -5 \end{cases}.$$

i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

ii) Determine the row-reduced echelon forms of the matrices $(A \mid b)$ and A and calculate their ranks.

iii) Find all the solutions to the linear system.

iv) Determine all $x \in \mathbb{R}^4$ which satisfy $Ax = b$ and which are orthogonal to the vector $u = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$.

2) (8 Points) Let $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$ and define the following three functions:

$$f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_1 + 3x_2 \\ x_1 + (u \bullet u)x_3 \end{pmatrix},$$

$$f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \sin(x_1) + \cos(x_2),$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \quad x \longmapsto \begin{pmatrix} x \bullet x \\ 0 \\ u \bullet u \end{pmatrix}.$$

i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.

ii) Is f_2 injective and/or surjective?

3) (8 Points)

i) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad G\begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

Determine the matrix of G.

ii) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map with

$$F\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, \quad F\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 6 \end{pmatrix}.$$

Show that F is not injective.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 - x_3 \\ x_1 + 2x_2 \\ x_2 + x_3 \end{pmatrix}.$$

i) Calculate the image of H.

ii) Decide if H is injective and/or surjective.

iii) Find all vectors $x \in \mathbb{R}^3$ with $H(x) = 2x$.

After finishing this exam submit your solution as one pdf file at NUCT at the "Midterm" assignment.

1) (10 Points) Consider the following linear system

$$\left\{\begin{array}{rcrcrcrcl} x_1 & + & 3x_2 & & & + & x_4 & = & 1 \\ & & x_2 & + & 2x_3 & - & 2x_4 & = & 2 \\ 2x_1 & - & 2x_2 & + & x_3 & + & x_4 & = & 3 \end{array}\right. .$$

i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

ii) Determine the row-reduced echelon forms of the matrices $(A \mid b)$ and A and calculate their ranks.

iii) Find all the solutions to the linear system.

iv) Determine all $x \in \mathbb{R}^4$ which satisfy $Ax = b$ and which have norm $\|x\| = \sqrt{14}$.

2) (8 Points) Let $u = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \mathbb{R}^2$ and define the following three functions:

$$f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} (u \bullet u) - 2 \\ x_1 + (u \bullet u)x_3 \end{pmatrix},$$

$$f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto e^{x_1} - e^{x_2},$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad x \longmapsto (x \bullet u)u .$$

i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.

ii) Is f_2 injective and/or surjective?

3) (8 Points) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}, \quad G\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}.$$

i) Determine the matrix of G.

ii) Find all vectors $x \in \mathbb{R}^2$ such that x is orthogonal to $G(x)$.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^4, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_1 + 2x_2 + x_3 \\ 2x_1 - 2x_3 \end{pmatrix}.$$

i) Calculate the image of H.

ii) Decide if H is injective and/or surjective.

iii) Find a linear map $F : \mathbb{R}^2 \to \mathbb{R}^4$ with $\mathrm{im}(F) = \mathrm{im}(H)$.

1) (10 Points) Consider the following linear system

$$\begin{cases} -x_1 + 2x_2 + 3x_3 + 2x_4 = 3 \\ 3x_1 - 2x_2 - x_3 + 2x_4 = 3 \\ x_1 + 2x_2 + 5x_3 + 6x_4 = 9 \end{cases}.$$

(i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

(ii) Calculate the row-reduced echelon forms of the matrices $(A \mid b)$ and A and calculate their ranks.

(iii) Determine all the solutions to the linear system $Ax = b$.

(iv) Find an injective linear map $F : \mathbb{R}^2 \to \mathbb{R}^4$ such that $Ax = 0$ for any $x \in \mathrm{im}(F)$.

2) (8 Points) Let $u = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \in \mathbb{R}^2$ and define the following three functions:

$$f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1(u \bullet u) - x_2 \\ x_1 + x_3 \end{pmatrix},$$

$$f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto x_1 \sin(x_2),$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad x \longmapsto (x \bullet u)x.$$

(i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.

(ii) Is f_2 injective and/or surjective?

3) (8 Points) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \quad G\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

(i) Determine the matrix of G.

(ii) Find all vectors $x \in \mathbb{R}^2$ such that x is orthogonal to every vector $v \in \mathrm{im}(G)$.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^4 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{pmatrix}.$$

(i) Calculate the image of H.

(ii) Decide if H is injective and/or surjective.

(iii) Find a linear map $J : \mathbb{R}^3 \to \mathbb{R}^4$ with $H(J(y)) = y$ for all $y \in \mathbb{R}^3$.

(iv) Show that there cannot exist a linear map $K : \mathbb{R}^3 \to \mathbb{R}^4$ with $K(H(x)) = x$ for all $x \in \mathbb{R}^4$.