Homework 4: Reflection, Projection and Inverses

Deadline: 10th December, 2024

Exercise 1. (2+3 = 5 Points) Let $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{R}^n$ be with $u \neq 0$.

(i) Show that the reflection $\rho_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear map.

(ii) Show that the matrix of the projection $P_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is given by

$$[P_u] = \frac{1}{u \bullet u} u u^T \in \mathbb{R}^{n\times n}\,,$$

where $u^T = (u_1\, u_2\, \ldots\, u_n) \in \mathbb{R}^{1\times n}$. Use this to give an expression for $[\rho_u]$.

Exercise 2. (2+3 = 5 Points) Let $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $d = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $x = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

(i) Calculate the matrices $[P_u]$ and $[\rho_u]$ in this special case.

(ii) Calculate the following vectors and draw them in one picture together with u,d and x

$$P_u(x), \quad \rho_u(x), \quad (P_u \circ P_d)(x), \quad \mathrm{rot}_{\frac{\pi}{2}}(x), \quad (P_u \circ \mathrm{rot}_{\frac{\pi}{2}})(x), \quad (\mathrm{rot}_{\frac{\pi}{2}} \circ P_u)(x).$$

Exercise 3. (2+2 = 4 Points) Show that for all $u \in \mathbb{R}^n$ with $u \neq 0$ the projection P_u and the reflection ρ_u satisfy for all $x \in \mathbb{R}^n$ the following two properties:

(i) $P_u(P_u(x)) = P_u(x)$.

(ii) $\rho_u(\rho_u(x)) = x$.

Exercise 4. (3+3= 6 Points)

(i) Decide if the following two linear maps are invertible. Determine their inverses if they exist.

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3\,, \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_2 + 2x_3 \\ x_1 - 4x_2 + 6x_3 \\ x_2 + x_3 \end{pmatrix},$$

$$G : \mathbb{R}^3 \longrightarrow \mathbb{R}^3\,, \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 4x_1 + x_2 - 2x_3 \\ x_1 - 2x_2 - 4x_3 \\ -x_1 + x_3 \end{pmatrix}.$$

(ii) The **kernel** of a linear map $H : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$\ker(H) = \{x \in \mathbb{R}^n \mid H(x) = 0\}\,.$$

Determine $\ker(F)$ and $\ker(G)$.

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くま先生の 簡単数学用語 解説コーナー

Hello ∼ クマ先生 here. I hope the midterms went well for all of you.

This week, we have four words relating to the subject of linear maps:

幾何(きか)　射影(しゃえい)　回転(かいてん)　鏡映(きょうえい)

These four words are: kika (**geometry**), shaei (**projection**), kaiten (**rotation**), and kyouei (**reflection**). To describe a linear map, one must add 変換(へんかん) after the word. For example, the reflection linear map is 鏡映変換(きょうえいへんかん) and the projection linear map is 射影変換(しゃえいへんかん).

Anyway, now, a breakdown of the individual (new) 漢字(かんじ) that makes up these words:

幾(き) - Commonly read as "いく". This kanji means "**(how) much**" or "**countless**". While uncommonly used (as these words are usually written in hiragana), this kanji is used in the word 幾(いく)つ (how much) and 幾(いく)ら (how much (price)).

何(か) - Commonly read as "なに", this kanji means "**what**". This kanji is used very commonly in everyday life, one of them being 何(なに) (meaning "what?").

影(えい) - This kanji means "**shadow**". It refers to how a projection is basically an "image" (or shadow) of something when projected onto another thing. One word that includes this kanji that might be familiar is 火影(ほかげ) (Hokage).

回(かい) - This kanji means "**(to) turn**". It refers to how rotations are.. things turning! A common use of this kanji is 回(まわ)る (to turn).

転(てん) - This kanji means "**(to) turn**". Again, it refers to how rotation turns things around. Common uses of this kanji are 自転車(じてんしゃ) (bicycle) and 転(ころ)ぶ (to tumble).

鏡(きょう) - Commonly read as かがみ, this kanji means "**mirror**". It refers to how reflection, in a sense, requires a mirror (or a mirror plane). One common use of this kanji is in 眼鏡(めがね) (Spectacles).

映(えい) - This kanji means "**(to) project (light)**". It refers to how reflection can be thought of as "projecting" through a mirror. The most common use of this kanji is 映画(えいが) (movie).

And that's it for today's (Mathematical) Japanese word(s). またね ∼