

## Homework 2: Matrices & Vectors

Deadline: 5th November, 23:55, 2024

**Exercise 1.** (2+2 = 4 Points) Show that for all  $A \in \mathbb{R}^{m \times n}$ ,  $x, y \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$  we have

- (i)  $A(x + y) = Ax + Ay$ ,
- (ii)  $A(\lambda x) = \lambda(Ax)$ .

(Without using Proposition 2.4. from the lecture).

Let  $A \in \mathbb{R}^{m \times n}$  be a matrix. The **rank**  $\text{rk}(A)$  of  $A$  is the number of pivot elements in  $\text{rref}(A)$ .  
For example,

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} = \text{rref}(A)$$

and therefore the rank of  $A$  is  $\text{rk}(A) = 2$ .

**Exercise 2.** (2+1+3 = 6 Points) Let  $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$  be a polynomial of degree 3 with real coefficients  $a_0, a_1, a_2, a_3 \in \mathbb{R}$ . For such a polynomial  $p$  define the vector  $v_p$  by

$$v_p = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^4.$$

- (i) Find a matrix  $D \in \mathbb{R}^{4 \times 4}$ , such that  $v_q = Dv_p$ , where  $q(x) = p(x) + 2p'(x)$ .  
(Here  $p'$  denotes the derivative of the polynomial  $p$  with respect to  $x$ ).
- (ii) Determine the rank of  $D$  in (i).
- (iii) For an arbitrary polynomial  $p$  of degree 3, find a polynomial  $s$ , such that  $v_p = Dv_s$ .

**Exercise 3.** (2+2 = 4 Points) Let  $A \in \mathbb{R}^{3 \times 3}$  be a matrix.

- (i) Show that if  $\text{rk}(A) = 3$  then there exists just one vector  $x \in \mathbb{R}^3$  with  $Ax = 0$ .
- (ii) Show that if  $\text{rk}(A) \leq 2$  then there exist infinitely many vectors  $x \in \mathbb{R}^3$  with  $Ax = 0$ .

**Exercise 4.** (4+2 = 6 Points) Let  $a, b, c, d \in \mathbb{R}$  and  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ .

- (i) Show that  $\text{rk}(A) = 2$  if and only if  $ad - bc \neq 0$ .
- (ii) We define the following subset of  $\mathbb{R}^2$

$$L = \{x \in \mathbb{R}^2 \mid x = Av \text{ for some } v \in \mathbb{R}^2\}.$$

How does  $L$  look like if  $\text{rk}(A) = 1$ ? How does it look like if  $\text{rk}(A) = 2$ ?

くま先生の  
**簡単数学用語**  
**解説コーナー**



Hello ~ クマ先生 here. With another homework comes another Japanese lesson~

This week, I have three words prepared for this time:

ぎょうれつ                      ほうていしき                      ベクトル  
**行列**                              **方程式**

These three words are: gyouretsu (**matrix**), houteishiki (**equation**), and bekutoru (**vector**). While the Japanese word for vector is a direct transliteration of the English word vector, the other two are not so.

The first word (行列) also has a meaning in everyday life: a procession (e.g. a wedding procession).

The second word (方程式), however, appears more in the sciences. Famous examples include Schrödinger's Equation (シュレディンガー方程式) and Maxwell's Equations (マクスウェルの方程式). Generally, however, this word can be used to describe any type of equation. For example, 行列方程式 (Matrix Equation), 代数方程式 (Algebraic Equation), ベクトル方程式 (Vector Equation) and 線型方程式 (Linear Equation).

Anyway, now, a breakdown of the individual 漢字 that makes up these two words:

ぎょう  
**行**

- This kanji means "row(s)". It refers to how matrices have "rows". In fact, the "row" of a matrix is called a 行 in Japanese. In everyday life, this kanji very common, as it is the kanji used in 行きます (meaning "to go" in Japanese).

れつ  
**列**

- This kanji means "column(s)". As such, the Japanese word for matrix literally means "rows of columns" or "columns of rows", which is an apt description of what a matrix is. In everyday life, this kanji shows up because of its other meaning : a queue (i.e. a queue in front of a shop).

ほう  
**方**

- This kanji means "direction" or "way". It refers to how an equation gives direction(s) on how the variables are related to one another. This kanji appears in a few words in everyday life, including 方法 (method), (あの) 方 (that person), and 貴方 (you; kanji literally means "precious person").

てい  
**程**

- This kanji means "about". In a sense, 方程式 (Equation) tells us something about some natural phenomenon. This kanji is uncommon in everyday life.

しき  
**式**

- This kanji means "formula" or "ceremony". This kanji can be found in any ceremony in everyday life, such as 卒業式 (Graduation Ceremony), 入学式 (Entrance ceremony), and 結婚式 (Wedding ceremony), among others.

And that's it for today's (Mathematical) Japanese word(s). またね ~