1) (12 Points) Let $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 & -1 \\ 2 & 0 & -3 \\ 3 & 2 & -3 \end{pmatrix}$.

(i) Determine whether or not the matrices A, B are invertible and, if they are, compute their inverses.

(ii) Give a basis for $\ker(AB)$ and $\ker(BA)$.

2) (14 Points) We define the subspace $U = \mathrm{span}\{u_1, u_2, u_3, u_4\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -2 \\ -6 \\ 4 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix}, \quad u_4 = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix}.$$

(i) Determine a basis $B = (b_1, \dots, b_m)$ of U and calculate its dimension.

(ii) Calculate the coordinate vectors $[u_1]_B$, $[u_2]_B$,$[u_3]_B$, and $[u_4]_B$, where B is the basis from (i).

(iii) Find a linear map $F : \mathbb{R}^4 \to \mathbb{R}^3$ with $\mathrm{im}(F) = U^\perp$ and determine $\dim(\ker(F))$.

3) (12 Points) Set $u = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$ and let $P : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Which of the following sets are subspaces? Justify your answers.

(i) $U_1 = \left\{ x \in \mathbb{R}^2 \mid x \bullet u + 5 = u \bullet u \right\}$.

(ii) $U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid x_1 - x_2 = 3x_3 \cdot x_2 \right\}$.

(iii) $U_3 = \{ x \in \mathbb{R}^n \mid P(x) = P(P(x)) \}$.

(iv) $U_4 = \left\{ x \in \mathbb{R}^2 \mid uu^T x = u \right\}$.

4) (12 Points) We consider the vector $b = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, the linear map

$$G : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 3 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and define the subspace $U = \mathrm{im}(G)$.

(i) Show that $\dim(U) = 2$ and find an orthonormal basis $F = (f_1, f_2)$ of U.

(ii) Determine $x \in \mathbb{R}^2$ such that $\|G(x) - b\|$ is minimal.

(iii) Determine the orthogonal projection $y = P_U(b)$ of b onto U and calculate $[y]_F$.