Tutorial 6: Review for the midterm exam

General comments:

(i) Next week Friday (17th November) we will have the midterm exam. Content: Lecture 1-5.

(ii) On the homepage you can now find the midterms + solutions of the last four years.

(iii) For Homework 3, Exercise 1 the videos "Image of a non-linear map" and "When is a linear map surjective/injective" might be helpful. (On the homepage at "Additional materials")

(iv) On Saturday 18th November we will have Lecture 6. You can choose if you want to do it face-to-face in the classroom or online in Zoom (which will be recorded). I will do a voting in the next Lecture.

Exercise 1. Consider the linear map

$$\begin{aligned} F : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ x &\longmapsto \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 2 \\ 1 & 1 & -5 \end{pmatrix} x \,. \end{aligned}$$

(i) Calculate $\operatorname{im}(F)$.

(ii) Is F surjective and/or injective?

(iii) Find all solutions to $F(x) = 0$.

(iv) Find all $x \in \mathbb{R}^3$ such that $v \bullet x = 0$ for all $v \in \operatorname{im}(F)$.

In (iv) the **dot product** $\bullet$ of $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$ is defined by

$$u \bullet v = u_1 v_1 + \cdots + u_n v_n \,.$$

This will be introduced in Lecture 5 (Friday 10th November).

The solutions for Exercises 2-7 can be found in "Tutorial 6 2019" on the homepage.

Exercise 2. Give an example of a linear system which has

(i) exactly one solution.

(ii) infinitely many solutions.

(iii) no solutions.

Exercise 3. Which of the following matrices are on row-reduced echelon form?

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Exercise 4. Consider the following linear system

$$\begin{cases} x_1 + 4x_2 + 7x_3 + 2x_4 = 1 \\ 2x_1 + 5x_2 + 8x_3 + x_4 = 2 \\ 3x_1 + 6x_2 + 10x_3 + x_4 = 1 \end{cases}.$$

(i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

(ii) Calculate the row-reduced echelon form of $(A \mid b)$.

(iii) Find all the solutions to the linear system.

(iv) Calculate the rank of $(A \mid b)$ and A.

Exercise 5. Define for a matrix $A \in \mathbb{R}^{m\times n}$ the linear map

$$\begin{aligned} F_A : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ x &\longmapsto Ax\,. \end{aligned}$$

Choose for A the matrices appearing in Exercise 2 & 3 and decide for each case if F_A is injective and/or surjective and calculate the image of F_A.

Exercise 6.

(i) Determine all vectors that are orthogonal to the vector $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(ii) Find a linear map $G : \mathbb{R}^2 \to \mathbb{R}^3$, such that the image of G is given by all the vectors which are orthogonal to v. (i.e. the image gives all the vectors you determined in i)).

Exercise 7. Let $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$ and define the following four functions:

$$\begin{aligned} f_1 : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \mathrm{rot}_x(u)\,, \end{aligned} \qquad \begin{aligned} f_2 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \begin{pmatrix} u \bullet x \\ (u \bullet u)(x \bullet u) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} f_3 : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &\longmapsto \begin{pmatrix} x_1 + x_2 + x_3 \\ 1 \end{pmatrix}, \end{aligned} \qquad \begin{aligned} f_4 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\longmapsto \begin{pmatrix} 2x_2 - x_1 \\ 2x_1 \\ 4x_1 + x_2 \end{pmatrix}. \end{aligned}$$

(i) Give a geometric interpretation of the image of f_1. Is f_1 injective and/or surjective?

(ii) Which of the above functions f_1, f_2, f_3, f_4 are linear maps? For each one that is linear, determine its matrix.

Exercise 1. Consider the linear map

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$x \longmapsto \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 2 \\ 1 & 1 & -5 \end{pmatrix} x\,.$$

i) Calculate $\mathrm{im}(F)$.

ii) Is F surjective and/or injective?

iii) Find all solutions to $F(x) = 0$.

iv) Find all $x \in \mathbb{R}^3$ such that $v \bullet x = 0$ for all $v \in \mathrm{im}(F)$.

i) We need to find all $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ s.th $F(x) = y$ has a solution.

$$([F]\,|\,y) = \left(\begin{array}{ccc|c} 1 & 0 & -1 & y_1 \\ 2 & -1 & 2 & y_2 \\ 1 & 1 & -5 & y_3 \end{array}\right) \xrightarrow{(-2),\,(-1)} \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & y_1 \\ 0 & -1 & 4 & y_2 - 2y_1 \\ 0 & 1 & -4 & y_3 - y_1 \end{array}\right) \xrightarrow{(-1),\,(1)}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & y_1 \\ 0 & 1 & -4 & 2y_1 - y_2 \\ 0 & 0 & 0 & -3y_1 + y_2 + y_3 \end{array}\right).$$

We get $\mathrm{im}(F) = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, -3y_1 + y_2 + y_3 = 0 \right\}$

ii) • $\mathrm{im}(F) \neq \mathbb{R}^3$ and therefore F is not surjective.

• We have $F\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = F\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ and therefore F is not injective.

Do iii) first and set $t=0$ and $t=1$

iii) By i) we see that for $y = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ the solutions of $F(x) = y = 0$ are given by $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ with

$$\begin{aligned} x_1 &= t \\ x_2 &= 4t \qquad t \in \mathbb{R} \\ x_3 &= t . \end{aligned}$$

iv) By i) we see that all vectors v the image of F have the form

$$v = \begin{pmatrix} y_1 \\ y_2 \\ 3y_1 - y_2 \end{pmatrix}$$

for any $y_1, y_2 \in \mathbb{R}$.

So for $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ we have

$$\begin{aligned} x \cdot v &= x_1 y_1 + x_2 y_2 + x_3 (3y_1 - y_2) \\ &= (x_1 + 3x_3) y_1 + (x_2 - x_3) y_2 . \end{aligned}$$

In order to have $x \cdot v$ for <u>all</u> $v \in \operatorname{im}(F)$, we therefore need to have $x_1+3x_3=0$ and $x_2-x_3=0$. This gives a linear system.

$$\begin{cases} x_1 \qquad + 3x_3 = 0 \\ \qquad x_2 - x_3 = 0 \end{cases}$$

Solutions: $\begin{aligned} x_1 &= -3t \\ x_2 &= t \\ x_3 &= t \end{aligned}$ for $t \in \mathbb{R}$

$\Longrightarrow$ All multiples of $\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$ are orthogonal to the vectors in the image of F.