Tutorial 5: Linear maps

A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear map**, if for all $u, v \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ we have

(i) $F(u + v) = F(u) + F(v)$,

(ii) $F(\lambda u) = \lambda F(u)$.

Exercise 1. Which of the following functions are linear maps?

$$\begin{aligned} f_1 : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \sin(x)\,, \end{aligned} \qquad \begin{aligned} f_2 : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^2 + 1\,, \end{aligned}$$

$$\begin{aligned} f_3 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\longmapsto \begin{pmatrix} 8x_1 + 2x_2 \\ 4x_2 \end{pmatrix}, \end{aligned} \qquad \begin{aligned} f_4 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\longmapsto \begin{pmatrix} x_1 + 2x_2 \\ x_1^2 + x_2 \end{pmatrix}. \end{aligned}$$

Theorem 4.2: For any linear map $F : \mathbb{R}^n \to \mathbb{R}^m$ there exist a unique matrix $[F] \in \mathbb{R}^{m\times n}$, such that we have for all $x \in \mathbb{R}^n$

$$F(x) = [F]x\,.$$

(Here the left-hand side is the evaluation of the function F at x and the right-hand side is the multiplication of the matrix $[F]$ with x.)

$[F]$ is called **the matrix of** F.

Conversely, we also saw that for any matrix $A \in \mathbb{R}^{m\times n}$ we can define a linear map $F : \mathbb{R}^n \to \mathbb{R}^m$ by setting $F(x) = Ax$. This is always a linear map (Example 16 in the lecture) whose matrix is $[F] = A$.

Exercise 2. Show that there exist a unique linear map $G : \mathbb{R}^2 \to \mathbb{R}^3$ with the property

$$G\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad G\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}.$$

What is the value of $G(x)$ for an arbitrary $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$? Determine the matrix of G.

Homework 3: Functions & Linear maps

Deadline: 12th November, 2023

Exercise 1. (3+3+4=10 Points) We define the following four functions:

$$f_1 : \mathbb{R} \longrightarrow \mathbb{R}^2, \quad x \longmapsto \begin{pmatrix} 1-\cos(x) \\ \sin(x) \end{pmatrix},$$

$$f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_1 - x_2 \\ x_1 x_2 \end{pmatrix},$$

$$f_3 : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{4x}{x^2+4},$$

$$f_4 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 3x_1 - x_2 \\ 2x_1 - x_2 \\ x_1 - x_2 \end{pmatrix}.$$

(i) Calculate the image of each function, i.e. describe $\mathrm{im}(f_j)$ for $j = 1, 2, 3, 4$ as explicit as possible. If you can not find a mathematical description try to describe the elements of the image in words.

(ii) Decide for each function if it is injective and/or surjective and/or bijective.

(iii) Decide which of the above functions are linear maps.

Justify your answers.

Exercise 2. (5 Points) Show that there exist a unique linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with the property

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

What is the value of $T(x)$ for an arbitrary $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$? Determine the matrix of T.

Exercise 3. (2+2+3=7 Points)

(i) Let X be a finite set. Show that a function $f : X \to X$ is injective if and only if it is surjective.

(ii) Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map. Show that F can not be surjective.

(iii) Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that F is injective if and only if the only solution to $F(x) = 0$ is $x = 0$.

Tutorial solutions

Exercise 1. Which of the following functions are linear maps?

$$f_1 : \mathbb{R} \longrightarrow \mathbb{R} \qquad x \longmapsto \sin(x)\,,$$

$$f_2 : \mathbb{R} \longrightarrow \mathbb{R} \qquad x \longmapsto x^2 + 1\,,$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 8x_1 + 2x_2 \\ 4x_2 \end{pmatrix}\,,$$

$$f_4 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + 2x_2 \\ x_1^2 + x_2 \end{pmatrix}\,.$$

1) f_1 is not linear since $\sin(\frac{\pi}{2}) = 1$, but $\sin(2\cdot\frac{\pi}{2}) = \sin(\pi) = 0 \neq 2\cdot\sin(\frac{\pi}{2}) = 2$.

Therefore $\lambda \sin(u) = \sin(\lambda u)$ is not true for all $\lambda \in \mathbb{R}$, $u \in \mathbb{R}^1$.

2) f_2 is not linear since $f_2(0+0) = f_2(0) = 1$, but $f_2(0) + f_2(0) = 1 + 1 = 2 \neq 1$.

3) f_3 is linear.

There are two ways to check this:

1. check the two conditions in the definition.

easier → 2. show that there exist a matrix $A \in \mathbb{R}^{2\times2}$ with $f_3(x) = Ax$

②: If we set $A = \begin{pmatrix} 8 & 2 \\ 0 & 4 \end{pmatrix}$, we have $f_3(x) = Ax$ and therefore f_3 is linear.

<u>or</u> : ①: For $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ we have

$$f_3(u+v) = f_3\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \end{pmatrix} = \begin{pmatrix} 8(u_1+v_1) + 2(u_2+v_2) \\ 4(u_2+v_2) \end{pmatrix}$$

$$= \begin{pmatrix} 8u_1+2u_2 + 8v_1+2v_2 \\ 4u_2 + 4v_2 \end{pmatrix} = \begin{pmatrix} 8u_1+2u_2 \\ 4u_2 \end{pmatrix} + \begin{pmatrix} 8v_1+2v_2 \\ 4v_2 \end{pmatrix}$$

$$= f_3(u) + f_3(v)$$

For $\lambda \in \mathbb{R}$:

$$f_3(\lambda u) = f_3\begin{pmatrix} \lambda u_1 \\ \lambda u_2 \end{pmatrix} = \begin{pmatrix} 8\lambda u_1 + 2\lambda u_2 \\ 4\lambda u_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda\,(8u_1+2u_2) \\ \lambda\,(4u_2) \end{pmatrix} = \lambda \begin{pmatrix} 8u_1+2u_2 \\ 4u_2 \end{pmatrix}$$

$$= \lambda f_3(u).$$

You see that in this case ① is much longer than ②!

Exercise 2. Show that there exist a unique linear map $G : \mathbb{R}^2 \to \mathbb{R}^3$ with the property

$$G\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}, \qquad G\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-1\\1\\-1\end{pmatrix}.$$

What is the value of $G(x)$ for an arbitrary $x = \begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{R}^2$? Determine the matrix of G.

If G is linear, then $G(u+v) = G(u) + G(v)$ and $G(\lambda u) = \lambda G(u)$. In particular, if

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}1\\2\end{pmatrix}, \text{ then}$$

$$G\begin{pmatrix}x_1\\x_2\end{pmatrix} = G\left(a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}1\\2\end{pmatrix}\right)$$

$$= aG\begin{pmatrix}1\\1\end{pmatrix} + bG\begin{pmatrix}1\\2\end{pmatrix} = a\begin{pmatrix}1\\0\\1\end{pmatrix} + b\begin{pmatrix}-1\\1\\-1\end{pmatrix}$$

$$= \begin{pmatrix}a-b\\b\\a-b\end{pmatrix}.$$

To find a, b we want to solve

$$\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$$

i.e. (as always) consider the augmented matrix

$$\begin{pmatrix} 1 & 1 & | & x_1 \\ 1 & 2 & | & x_2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & | & x_1 \\ 0 & 1 & | & x_2 - x_1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & | & 2x_1 - x_2 \\ 0 & 1 & | & x_2 - x_1 \end{pmatrix}$$

$\overset{\text{unique solution}}{\Longrightarrow}$ $a = 2x_1 - x_2, \quad b = x_2 - x_1.$

Therefore for any $x_1, x_2 \in \mathbb{R}$ we have

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (2x_1 - x_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (x_2 - x_1) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(as you can check by writing out the right-hand side).

In particular, if G is a linear map, then

$$G\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (2x_1 - x_2)\, G\begin{pmatrix} 1 \\ 1 \end{pmatrix} + (x_2 - x_1)\, G\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$= (2x_1 - x_2) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + (x_2 - x_1) \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3x_1 - 2x_2 \\ x_2 - x_1 \\ 3x_1 - 2x_2 \end{pmatrix}$$

Since $\begin{pmatrix} 3x_1 - 2x_2 \\ x_2 - x_1 \\ 3x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

we have $[G] = \begin{pmatrix} 3 & -2 \\ -1 & 1 \\ 3 & -2 \end{pmatrix}$.