# Tutorial 5: Linear maps

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A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear map**, if for all $u, v \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ we have

(i) $F(u + v) = F(u) + F(v)$,

(ii) $F(\lambda u) = \lambda F(u)$.

**Exercise 1.** Which of the following functions are linear maps?

$$f_1 : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \sin(x)\,,$$

$$f_2 : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto x^2 + 1\,,$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 8x_1 + 2x_2 \\ 4x_2 \end{pmatrix}\,,$$

$$f_4 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + 2x_2 \\ x_1^2 + x_2 \end{pmatrix}\,.$$

__**Theorem 4.2:**__ For any linear map $F : \mathbb{R}^n \to \mathbb{R}^m$ there exist a unique matrix $[F] \in \mathbb{R}^{m \times n}$, such that we have for all $x \in \mathbb{R}^n$

$$F(x) = [F]x\,.$$

(Here the left-hand side is the evaluation of the function $F$ at $x$ and the right-hand side is the multiplication of the matrix $[F]$ with $x$.)

$[F]$ is called **the matrix of** $F$.

Conversely, we also saw that for any matrix $A \in \mathbb{R}^{m \times n}$ we can define a linear map $F : \mathbb{R}^n \to \mathbb{R}^m$ by setting $F(x) = Ax$. This is always a linear map (Example 16 in the lecture) whose matrix is $[F] = A$.

**Exercise 2.** Show that there exist a unique linear map $G : \mathbb{R}^2 \to \mathbb{R}^3$ with the property

$$G\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\,, \qquad G\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}\,.$$

What is the value of $G(x)$ for an arbitrary $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$? Determine the matrix of $G$.