Tutorial 3: Matrices & Vectors

Recall that a $m\times n$-**matrix** is given by an array (with m rows and n columns) of numbers $a_{ij} \in \mathbb{R}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = (a_{ij})_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} = (a_{ij})_{ij}\,.$$

By $\mathbb{R}^{m\times n} = M_{m\times n}(\mathbb{R})$ we denote the set all of all $m\times n$-matrices.
A (column-) **vector** of size n is a $n\times 1$-matrix

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

and the set of all vectors of size n is denoted by $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. For $A, B \in \mathbb{R}^{m\times n}$ and $\lambda \in \mathbb{R}$ we defined

$$\begin{aligned} A + B &= (a_{ij} + b_{ij})_{ij} \in \mathbb{R}^{m\times n} && \text{(Sum of two matrices)}\,, \\ \lambda A &= (\lambda a_{ij})_{ij} \in \mathbb{R}^{m\times n} && \text{(Scalar multiplication)}\,. \end{aligned}$$

In the case $\lambda = -1$ we write $(-1)A = -A$ and $A - B$ means $A + (-1)B$.

The product of a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $v \in \mathbb{R}^n$ is defined by

$$Av = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + \dots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \dots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \dots + a_{mn}v_n \end{pmatrix} \in \mathbb{R}^m \qquad \text{(Matrix-vector multiplication)}\,.$$

In other words we have: ($m\times n$-matrix) $\cdot$ (vector of size n) = (vector of size m).

Example: $m = 2$, $n = 3$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 1\cdot 7 + 2\cdot 8 + 3\cdot 9 \\ 4\cdot 7 + 5\cdot 8 + 6\cdot 9 \end{pmatrix} = \begin{pmatrix} 50 \\ 122 \end{pmatrix}.$$

Exercise 1. We define the following matrices and vectors:

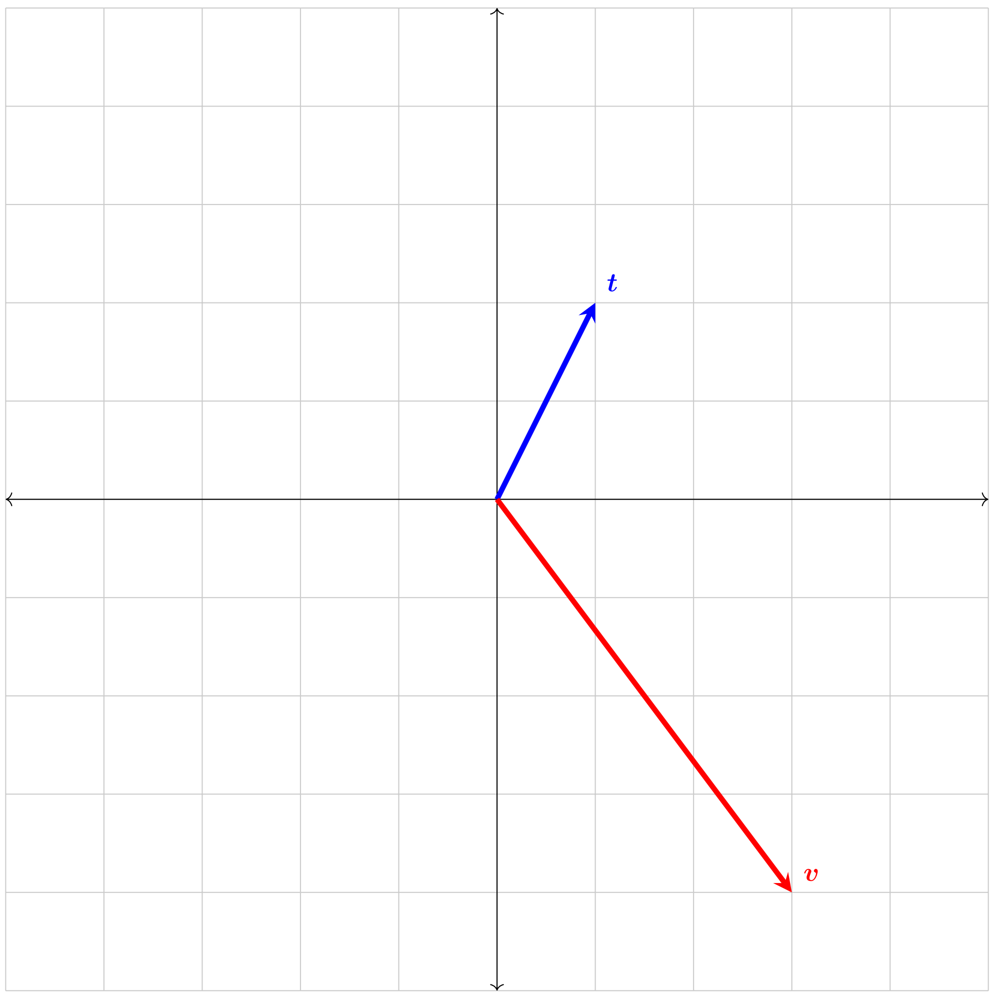
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 2 & 4 \\ 0 & -3 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 8 & 0 \\ 1 & 2 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix},$$

$$t = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad u = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad v = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \quad w = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}.$$

Decide which of the following expressions are defined. Evaluate them if possible.

$$At, \quad Au, \quad wA, \quad 2A, \quad A+B, \quad A+C, \quad A+D, \quad \frac{3}{4}Bt, \quad Bu, \quad B+B, \quad Dw,$$
$$Cv, \quad t+u, \quad tu, \quad -v, \quad u+w, \quad t-u, \quad \frac{1}{2}w, \quad C+w, \quad Et, \quad Ev \quad E\,(Ev)\,.$$

Exercise 2. The vectors t and v of Exercise 1 are drawn in the following diagram.



Draw the following vectors in the diagram:

$$-2t, \quad t-\frac{1}{2}v, \quad v+t, \quad t+v, \quad Et, \quad Ev, \quad E\,(Ev)\,, \quad Bt, \quad Bv\,.$$

Can you guess what happens in general to a vector in $\mathbb{R}^2$ when you multiply it with B or E?

Exercise 3.

i) Calculate the rank of the matrices in Exercise 1.

ii) Find a vector $x \in \mathbb{R}^2$, such that $Ex = t$ and draw it in the diagram above.

iii) Find a matrix $M \in \mathbb{R}^{2\times 2}$ with $Mv = t$. Is this matrix unique?