Tutorial 3&4

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Tutorial 3 &4: Matrices & Vectors

Recall that a $m\times n$-**matrix** is given by an array (with m rows and n columns) of numbers $a_{ij} \in \mathbb{R}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = (a_{ij})_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} = (a_{ij})_{ij}\,.$$

By $\mathbb{R}^{m\times n} = M_{m\times n}(\mathbb{R})$ we denote the set all of all $m\times n$-matrices.
A (column-) **vector** of size n is a $n\times 1$-matrix

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

and the set of all vectors of size n is denoted by $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. For $A, B \in \mathbb{R}^{m\times n}$ and $\lambda \in \mathbb{R}$ we defined

$$\begin{aligned} A + B &= (a_{ij} + b_{ij})_{ij} \in \mathbb{R}^{m\times n} && \text{(Sum of two matrices)}\,, \\ \lambda A &= (\lambda a_{ij})_{ij} \in \mathbb{R}^{m\times n} && \text{(Scalar multiplication)}\,. \end{aligned}$$

In the case $\lambda = -1$ we write $(-1)A = -A$ and $A - B$ means $A + (-1)B$.

The product of a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $v \in \mathbb{R}^n$ is defined by

$$Av = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + \dots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \dots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \dots + a_{mn}v_n \end{pmatrix} \in \mathbb{R}^m \qquad \text{(Matrix-vector multiplication)}\,.$$

In other words we have: ($m\times n$-matrix) $\cdot$ (vector of size n) = (vector of size m).

Example: $m = 2$, $n = 3$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 1\cdot 7 + 2\cdot 8 + 3\cdot 9 \\ 4\cdot 7 + 5\cdot 8 + 6\cdot 9 \end{pmatrix} = \begin{pmatrix} 50 \\ 122 \end{pmatrix}.$$

Exercise 1. We define the following matrices and vectors:

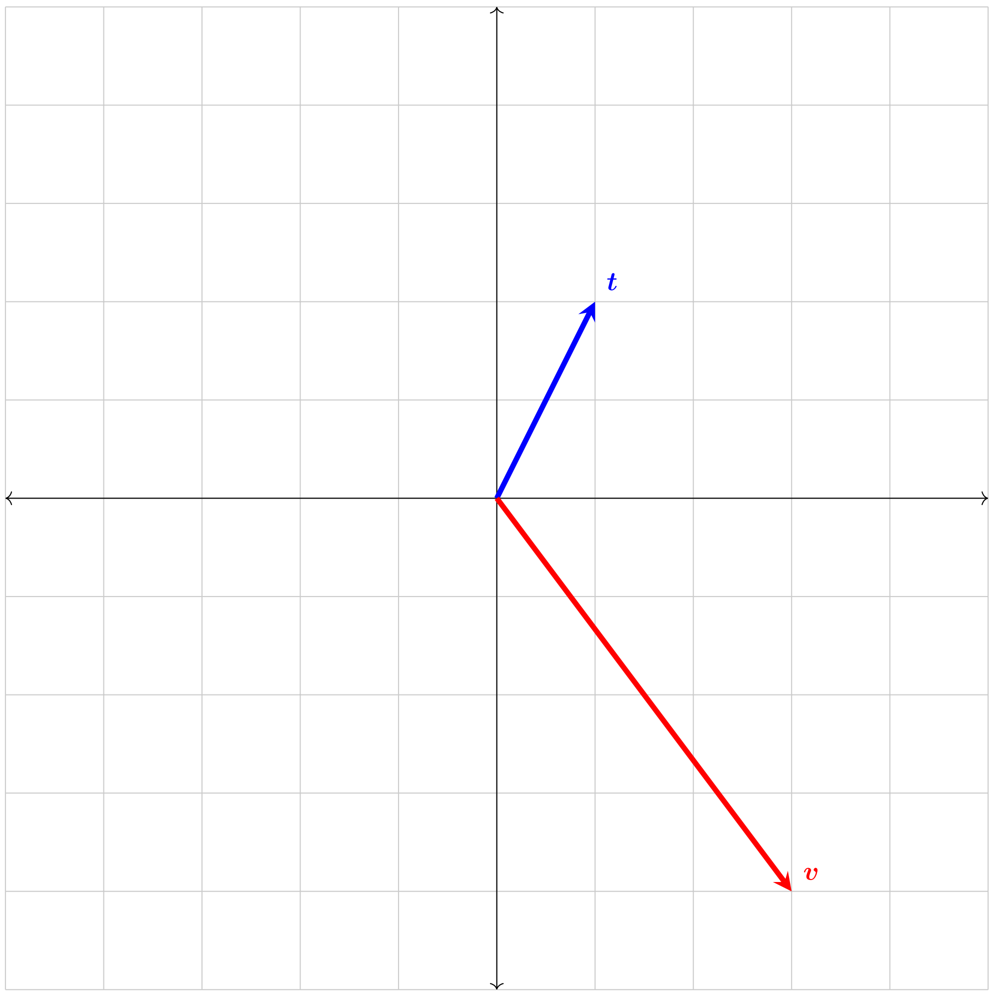
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 2 & 4 \\ 0 & -3 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 8 & 0 \\ 1 & 2 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix},$$

$$t = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad u = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad v = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \quad w = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}.$$

Decide which of the following expressions are defined. Evaluate them if possible.

$$At, \quad Au, \quad wA, \quad 2A, \quad A+B, \quad A+C, \quad A+D, \quad \frac{3}{4}Bt, \quad Bu, \quad B+B, \quad Dw,$$
$$Cv, \quad t+u, \quad tu, \quad -v, \quad u+w, \quad t-u, \quad \frac{1}{2}w, \quad C+w, \quad Et, \quad Ev \quad E\,(Ev)\,.$$

Exercise 2. The vectors t and v of Exercise 1 are drawn in the following diagram.



Draw the following vectors in the diagram:

$$-2t, \quad t-\frac{1}{2}v, \quad v+t, \quad t+v, \quad Et, \quad Ev, \quad E\,(Ev)\,, \quad Bt, \quad Bv\,.$$

Can you guess what happens in general to a vector in $\mathbb{R}^2$ when you multiply it with B or E?

Exercise 3.

i) Calculate the rank of the matrices in Exercise 1.

ii) Find a vector $x \in \mathbb{R}^2$, such that $Ex = t$ and draw it in the diagram above.

iii) Find a matrix $M \in \mathbb{R}^{2\times 2}$ with $Mv = t$. Is this matrix unique?

Homework 2: Matrices & Vectors

Deadline: 29th October, 23:55, 2023

Exercise 1. (2+2 = 4 Points) Show that for all $A \in \mathbb{R}^{m\times n}$, $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we have

(i) $A(x + y) = Ax + Ay$,

(ii) $A(\lambda x) = \lambda(Ax)$.

(Without using Proposition 2.4. from the lecture).

Let $A \in \mathbb{R}^{m\times n}$ be a matrix. The **rank** $\mathrm{rk}(A)$ of A is the number of pivot elements in $\mathrm{rref}(A)$.
For example,

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{rref}(A) \quad ,$$

and therefore the rank of A is $\mathrm{rk}(A) = 2$.

Exercise 2. (2+1+3 = 6 Points) Let $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ be a polynomial of degree 3 with real coefficients $a_0, a_1, a_2, a_3 \in \mathbb{R}$. For such a polynomial p define the vector v_p by

$$v_p = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^4 .$$

(i) Find a matrix $D \in \mathbb{R}^{4\times 4}$, such that $v_q = Dv_p$, where $q(x) = 2p(x) - p'(x)$.
(Here p' denotes the derivative of the polynomial p with respect to x).

(ii) Determine the rank of D in (i).

(iii) For an arbitrary polynomial p of degree 3, find a polynomial s, such that $v_p = Dv_s$.

Exercise 3. (2+2 = 4 Points) Let $A \in \mathbb{R}^{3\times 3}$ be a matrix.

(i) Show that if $\mathrm{rk}(A) = 3$ then there exists just one vector $x \in \mathbb{R}^3$ with $Ax = 0$.

(ii) Show that if $\mathrm{rk}(A) \leq 2$ then there exist infinitely many vectors $x \in \mathbb{R}^3$ with $Ax = 0$.

Exercise 4. (4+2 = 6 Points) Let $a, b, c, d \in \mathbb{R}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(i) Show that $\mathrm{rk}(A) = 2$ if and only if $ad - bc \neq 0$.

(ii) We define the following subset of $\mathbb{R}^2$

$$L = \{x \in \mathbb{R}^2 \mid x = Av \text{ for some } v \in \mathbb{R}^2\}.$$

How does L look like if $\mathrm{rk}(A) = 1$? How does it look like if $\mathrm{rk}(A) = 2$?

Solution to the Tutorial Exercises:

$$A=\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix},\quad B=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad C=\begin{pmatrix}-1&0\\2&4\\0&-3\end{pmatrix},\quad D=\begin{pmatrix}0&8&0\\1&2&-1\end{pmatrix},\quad E=\begin{pmatrix}\frac35&-\frac45\\\frac45&\frac35\end{pmatrix},$$

$$t=\begin{pmatrix}1\\2\end{pmatrix},\quad u=\begin{pmatrix}1\\-1\\2\end{pmatrix},\quad v=\begin{pmatrix}3\\-4\end{pmatrix},\quad w=\begin{pmatrix}-1\\2\\2\end{pmatrix}.$$

Decide which of the following expressions are defined. Evaluate them if possible.

$$\cancel{At},\quad Au,\quad \cancel{uA},\quad 2A,\quad \cancel{A+B},\quad \cancel{A+C},\quad A+D,\quad \frac{3}{4}Bt,\quad \cancel{Bu},\quad B+B,\quad Dw,$$

$$Cv,\quad \cancel{t+u},\quad \cancel{tu},\quad -v,\quad u+w,\quad \cancel{t\cdot u},\quad \frac{1}{2}w,\quad \cancel{C+w},\quad Et,\quad Ev\quad E(Ev).$$

$X =$ Not defined (Either wrong dimension or just not defined)

$Au = \begin{pmatrix}5\\11\end{pmatrix}$

$2A = \begin{pmatrix}2&4&6\\8&10&12\end{pmatrix}$

$A+D = \begin{pmatrix}1&10&3\\5&7&5\end{pmatrix}$

$\frac34 Bt = \begin{pmatrix}\frac32\\-\frac34\end{pmatrix}$

$B+B = \begin{pmatrix}0&2\\-2&0\end{pmatrix}$

$Dw = \begin{pmatrix}16\\1\end{pmatrix}$

$Cv = \begin{pmatrix}-3\\-10\\12\end{pmatrix}$

$-v = \begin{pmatrix}-3\\4\end{pmatrix}$

$u+w = \begin{pmatrix}0\\1\\4\end{pmatrix}$

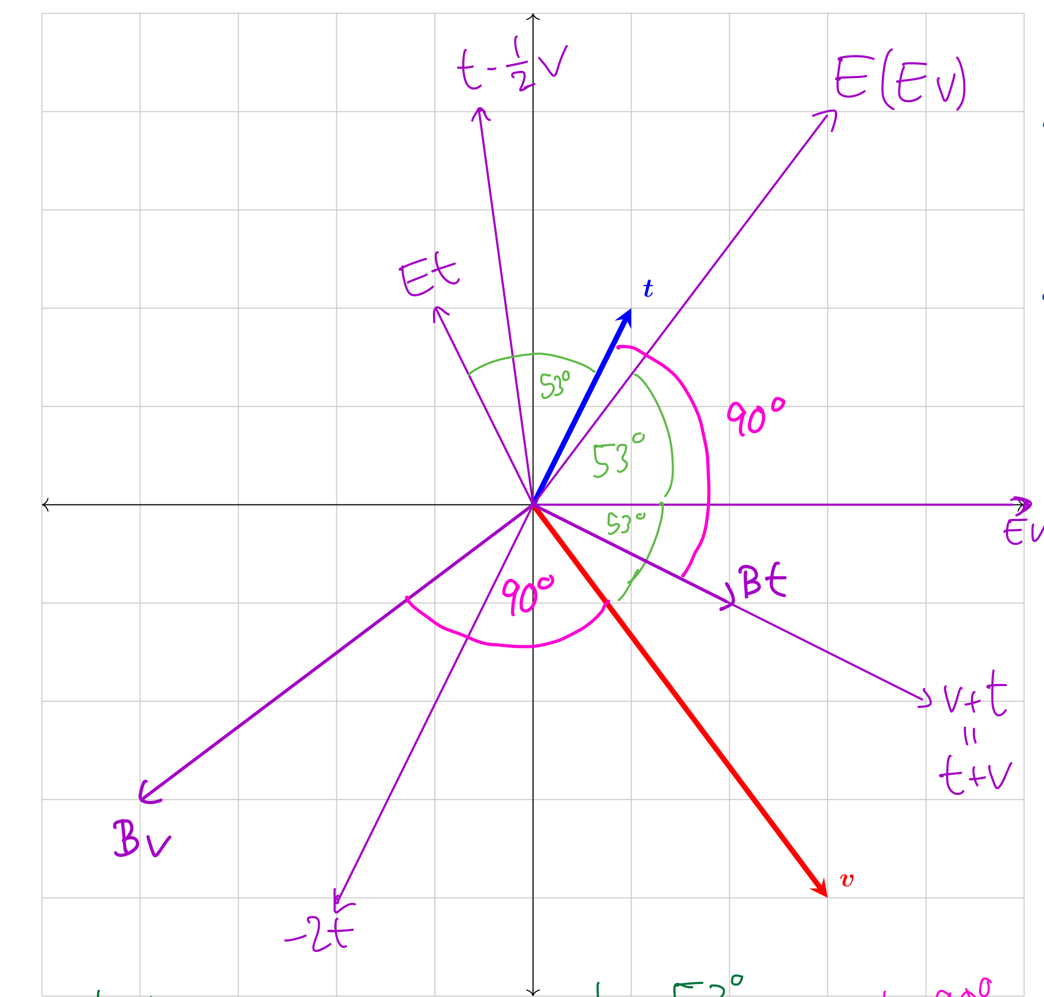
$\frac12 w = \begin{pmatrix}-\frac12\\1\\1\end{pmatrix}$

$Et = \begin{pmatrix}-1\\2\end{pmatrix}$

$Ev = \begin{pmatrix}5\\0\end{pmatrix}$

$E(Ev) = \begin{pmatrix}3\\4\end{pmatrix}$

Exercise 2. The vectors t and v of Exercise 1 are drawn in the following diagram.



Multiplying with E: Rotation by 53°

Mult. by B: Rotation by 90°

Draw the following vectors in the diagram:

$$-2t,\quad t-\frac{1}{2}v,\quad v+t,\quad t+v,\quad Et,\quad Ev,\quad E(Ev),\quad Bt,\quad Bv.$$

Can you guess what happens in general to a vector in $\mathbb{R}^2$ when you multiply it with B or E?

$-2t = \begin{pmatrix}-2\\-4\end{pmatrix}$

$t-\frac12 v = \begin{pmatrix}1\\2\end{pmatrix} - \begin{pmatrix}\frac32\\-2\end{pmatrix} = \begin{pmatrix}-\frac12\\4\end{pmatrix}$

Exercise 3.

i) Calculate the rank of the matrices in Exercise 1.

ii) Find a vector $x\in\mathbb{R}^2$, such that $Ex = t$ and draw it in the diagram above.

iii) Find a matrix $M\in\mathbb{R}^{2\times 2}$ with $Mv = t$. Is this matrix unique?

$Bt = \begin{pmatrix}2\\-1\end{pmatrix}$

$Bv = \begin{pmatrix}-4\\-3\end{pmatrix}$

3) i)

$$A=\begin{pmatrix}1&2&3\\4&5&6\end{pmatrix},\quad B=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad C=\begin{pmatrix}-1&0\\2&4\\0&-3\end{pmatrix},\quad D=\begin{pmatrix}0&8&0\\1&2&-1\end{pmatrix},\quad E=\begin{pmatrix}\frac{3}{5}&-\frac{4}{5}\\\frac{4}{5}&\frac{3}{5}\end{pmatrix},$$

$$A\sim\begin{pmatrix}1&2&3\\0&-3&-6\end{pmatrix}\sim\begin{pmatrix}1&2&3\\0&1&2\end{pmatrix}\sim\begin{pmatrix}1&0&-1\\0&1&2\end{pmatrix}=rref(A)$$

$$\Rightarrow\quad rk(A)=2$$

$$rk(B)=2,\quad rk(C)=2,\quad rk(D)=2$$

$$rk(E)=2$$

ii) Find $x\in\mathbb{R}^2$ with $Ex=t \iff \begin{pmatrix}\frac{3}{5}&-\frac{4}{5}\\\frac{4}{5}&\frac{3}{5}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$

$$(E|t)=\begin{array}{c}5\\5\end{array}\left(\begin{array}{cc|c}\frac{3}{5}&-\frac{4}{5}&1\\\frac{4}{5}&\frac{3}{5}&2\end{array}\right)\sim\begin{array}{c}-1\\ \end{array}\left(\begin{array}{cc|c}3&-4&5\\4&3&10\end{array}\right)$$

$$\sim\begin{array}{c}\\-3\end{array}\left(\begin{array}{cc|c}3&-4&5\\1&7&5\end{array}\right)\sim\left(\begin{array}{cc|c}0&-25&-10\\1&7&5\end{array}\right)$$

$$\sim\begin{array}{c}\\-\frac{1}{25}\end{array}\left(\begin{array}{cc|c}1&7&5\\0&-25&-10\end{array}\right)\sim\begin{array}{c}\\-7\end{array}\left(\begin{array}{cc|c}1&7&5\\0&1&\frac{2}{5}\end{array}\right)$$

$$\sim\left(\begin{array}{cc|c}1&0&\frac{11}{5}\\0&1&\frac{2}{5}\end{array}\right)\qquad \text{Solution: } x=\begin{pmatrix}\frac{11}{5}\\\frac{2}{5}\end{pmatrix}$$

iii) Find $M \in \mathbb{R}^{2\times 2}$ with $Mv = t$

$$\iff \quad M\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Set $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$M\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3a-4b \\ 3c-4d \end{pmatrix}$$

We therefore want to solve $\begin{pmatrix} 3a-4b \\ 3c-4d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

This is in fact a linear system

$$\begin{array}{c} \frac{1}{3} \\ \frac{1}{3} \end{array} \left\{ \begin{array}{l} 3a - 4b = 1 \\ 3c - 4d = 2 \end{array} \right.$$

free

$$\iff \left\{ \begin{array}{l} \underline{a} - \frac{4}{3}\underline{b} = \frac{1}{3} \\ \underline{c} - \frac{4}{3}\underline{d} = \frac{2}{3} \end{array} \right.$$

Pivot

Solutions: $\quad a = \frac{1}{3} + \frac{4}{3}t_1, \quad b = t_1 \quad , \quad c = \frac{2}{3} + \frac{4}{3}t_2, \quad d = t_2$

All matrices of the form

$$M = \begin{pmatrix} \frac{1}{3} + \frac{4}{3} t_1 & t_1 \\ \frac{2}{3} + \frac{4}{3} t_2 & t_2 \end{pmatrix}$$

for any $t_1, t_2 \in \mathbb{R}$ satisfy $Mv = t$.

In particular, such a matrix is not unique.

Comments to Homework 2 :

Exercise 1. (3+3 = 6 Points) Show that for all $A \in \mathbb{R}^{m \times n}$, $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we have

i) $A(x + y) = Ax + Ay$,

ii) $A(\lambda x) = \lambda(Ax)$.

Write $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ and $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

and then calculate both sides

$$A(x+y) = A \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix} = \ldots\ldots$$

$$\|$$

$$Ax + Ay = \ldots\ldots = \ldots$$

Exercise 2. (2+1+3 = 6 Points) Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ be a polynomial of degree 3 with real coefficients $a_0, a_1, a_2, a_3 \in \mathbb{R}$. For such a polynomial p define the vector v_p by

$$v_p = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^4.$$

(i) Find a matrix $D \in \mathbb{R}^{4\times4}$, such that $v_q = Dv_p$, where $q(x) = 2p(x) - p'(x)$. (Here p' denotes the derivative of the polynomial p with respect to x).

(ii) Determine the rank of D in (i).

(iii) For an arbitrary polynomial p of degree 3, find a polynomial s, such that $v_p = Dv_s$.

Example: $p(x) = 1 + 2x - 3x^2 + 5x^3$ then

$$v_p = \begin{pmatrix} 1 \\ 2 \\ -3 \\ 5 \end{pmatrix}$$

(i): • Calculate $2p(x) - p'(x)$ for $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

(ii) calculate rref(D)

(iii) Solve linear system.

Exercise 3. (2+2 = 4 Points) Let $A \in \mathbb{R}^{3\times 3}$ be a matrix.

(i) Show that if $\text{rk}(A) = 3$ then there exists just one vector $x \in \mathbb{R}^3$ with $Ax = 0$.

(ii) Show that if $\text{rk}(A) \leq 2$ then there exist infinitely many vectors $x \in \mathbb{R}^3$ with $Ax = 0$.

If $A \in \mathbb{R}^{3\times3}$ how can $\text{rref}\left(A \,\middle|\, \begin{smallmatrix}0\\0\\0\end{smallmatrix}\right)$ look like if $\text{rk}(A) = 3$ and $\text{rk}(A) \leq 2$? What does this say about the solutions of $Ax = 0$?

Exercise 4. (4+2 = 6 Points) Let $a, b, c, d \in \mathbb{R}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(i) Show that $\text{rk}(A) = 2$ if and only if $ad - bc \neq 0$.

(ii) We define the following subset of $\mathbb{R}^2$

$$L = \{x \in \mathbb{R}^2 \mid x = Av \text{ for some } v \in \mathbb{R}^2\}.$$

How does L look like if $\text{rk}(A) = 1$? How does it look like if $\text{rk}(A) = 2$?

Show two directions:

① $\text{rk}(A) = 2 \implies ad - bc \neq 0$

and ② $ad - bc \neq 0 \implies \text{rk}(A) = 2$

If P_1 and P_2 are statements then
$P_1 \Rightarrow P_2$
$\Leftrightarrow \neg P_2 \Rightarrow \neg P_1$
negate

For ①: Show that if $ad - bc = 0$ then $\text{rk}(A) \neq 2$

For ②:
Try to bring $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to ref. If $ad - bc \neq 0$ then you are allowed to divide by it.