Tutorial 2 10/10/2023

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Tutorial 2: Linear systems

Elementary row operations:.

(R1) Add a multiple of an equation to another.

(R2) Multiply an equation with a non-zero number.

(R3) Change the order of the equations.

Gaussian elimination / Row reduction: Given a linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

one procedure for bringing this linear system to its row-reduced echelon form is as follows:

I. Downwards:

1) Make the first equation contain the first variable by using (R3).
2) Make the coefficient of this variable equal to 1 by using (R2).
3) Eliminate this variable from all other equations by using (R1).
4) Iterate with the first occurring variable in the remaining equations.

II. Upwards

1) Let x_i be the first variable in the last equation. Eliminate x_i from all other equations by using (R1).
2) Go to previous equations and iterate.

Remark: The above algorithm always works, but in practice there are sometimes shorter / better ways to obtain the reduced echelon form.

Exercise 1. Which of the following linear systems are on row-reduced echelon form? For those that are not, find an equivalent system (i.e. one which has the same solutions) that is on row-reduced echelon form. For each system, find all solutions.

i) $\begin{cases} 3x_1 + 4x_2 + 7x_3 + 9x_4 = 5 \\ x_1 + x_2 - x_4 = 0 \end{cases}$

ii) $\{\, x_1 + x_2 + x_3 = 1$

Exercise 2. Decide for which real numbers $a \in \mathbb{R}$ the following linear system has solutions. Give all the solutions in these cases.

$$\begin{cases} 2x_2 + 2x_3 = 2a+8 \\ x_1 + x_2 - x_3 = 4 \\ 2x_1 - 4x_3 = a+9 \end{cases}.$$

rref

Exercise 1. Which of the following linear systems are on row-reduced echelon form? For those that are not, find an equivalent system (i.e. one which has the same solutions) that is on row-reduced echelon form. For each system, find all solutions.

i) $\begin{cases} 3x_1 + 4x_2 + 7x_3 + 9x_4 = 5 \\ x_1 + x_2 - x_4 = 0 \end{cases}$

ii) $\{ \; x_1 + x_2 + x_3 = 1$

i) This linear system is not on rref since the first non-zero coefficient is not 1.

$$\begin{array}{r} \\ (-3) \end{array}\begin{cases} 3x_1+4x_2+7x_3+9x_4=5 \\ x_1+x_2 \qquad\quad -x_4=0 \end{cases} \iff \begin{array}{r} \\ (-1) \end{array}\begin{cases} x_1+x_2 \qquad\quad -x_4=0 \\ \qquad x_2+7x_3+12x_4=5 \end{cases}$$

$$\iff \begin{cases} x_1 \qquad -7x_3-13x_4=-5 \\ \qquad x_2+7x_3+12x_4=5 \end{cases}$$

on rref

pivot   free

The variables x_3 and x_4 are free and we set $x_3=t_1, x_4=t_2$ for $t_1,t_2 \in \mathbb{R}$. All solutions are given by

$$\begin{aligned} x_1 &= -5+7t_1+13t_2 \\ x_2 &= 5-7t_1-12t_2 \\ x_3 &= t_1 \\ x_4 &= t_2 \end{aligned}$$

Remark: Notice the Gaussian elimination would give:

$$\begin{array}{r} (\tfrac{1}{3}) \\ \\ \end{array}\begin{cases} 3x_1+4x_2+7x_3+9x_4=5 \\ x_1+x_2 \qquad\quad -x_4=0 \end{cases} \iff \begin{array}{r} (-1) \\ \\ \end{array}\begin{cases} x_1+\frac{4}{3}x_2+\frac{7}{3}x_3+3x_4=\frac{5}{3} \\ x_1+x_2 \qquad\quad -x_4=0 \end{cases}$$

$$\iff \begin{array}{r} \\ (-3) \end{array}\begin{cases} x_1+\frac{4}{3}x_2+\frac{7}{3}x_3+3x_4=\frac{5}{3} \\ \quad -\frac{1}{3}x_2-\frac{7}{3}x_3-4x_4=-\frac{5}{3} \end{cases} \iff \begin{array}{r} \\ (-\frac{4}{3}) \end{array}\begin{cases} x_1+\frac{4}{3}x_2+\frac{7}{3}x_3+3x_4=\frac{5}{3} \\ \qquad x_2+7x_3+12x_4=5 \end{cases}$$

$$\Longleftrightarrow \begin{cases} x_1 \quad -7x_3 - 13x_4 = -5 \\ x_2 + 7x_3 + 12x_4 = 5 \end{cases}$$

This is the same result as before, but the calculation became more difficult.

ii) The linear system $\{x_1 + x_2 + x_3 = 1$ is on r.ref.
The variables x_2, x_3 are free and solutions are

$$\begin{aligned} x_1 &= 1 - t_1 - t_2 \\ x_2 &= t_1 \\ x_3 &= t_2 \end{aligned} \qquad t_1, t_2 \in \mathbb{R}.$$

Exercise 2. Decide for which real numbers $a \in \mathbb{R}$ the following linear system has solutions. Give all the solutions in these cases.

$$\begin{cases} \quad\;\; 2x_2 + 2x_3 = 2a+8 \\ x_1 + x_2 - x_3 = 4 \\ 2x_1 \qquad\; - 4x_3 = a+9 \end{cases}.$$

$$\left(\tfrac{1}{2}\right)\begin{cases} 2x_2+2x_3=2a+8 \\ x_1+x_2-x_3=4 \\ 2x_1 \qquad -4x_3=a+9 \end{cases} \iff (-2)\begin{cases} x_1+x_2-x_3=4 \\ x_2+x_3=a+4 \\ 2x_1 \qquad -4x_3=a+9 \end{cases}$$

$$\iff (2)\begin{cases} x_1+x_2-x_3=4 \\ x_2+x_3=a+4 \\ -2x_2-2x_3=a+1 \end{cases} \iff \begin{cases} x_1+x_2-x_3=4 \\ x_2+x_3=a+4 \\ 0=3a+9 \end{cases}$$

If $3a+9\neq 0$ (i.e. $a\neq -3$) there are no solutions.

If $a=-3$ the linear system becomes

$$(-1)\begin{cases} x_1+x_2-x_3=4 \\ x_2+x_3=1 \\ (0=0) \end{cases} \iff \begin{cases} x_1 \qquad -2x_3=3 \\ x_2+x_3=1 \end{cases}$$

Setting $x_3=t$ $(t\in\mathbb{R})$ we get the solution:

$$x_1=3+2t$$
$$x_2=1-t$$
$$x_3=t$$

$\Rightarrow$ For $a=-3$ there are infinitely many solutions.