Tutorial 2: Linear systems

Elementary row operations:.

(R1) Add a multiple of an equation to another.

(R2) Multiply an equation with a non-zero number.

(R3) Change the order of the equations.

Gaussian elimination / Row reduction: Given a linear system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

one procedure for bringing this linear system to its row-reduced echelon form is as follows:

I. Downwards:

1) Make the first equation contain the first variable by using (R3).
2) Make the coefficient of this variable equal to 1 by using (R2).
3) Eliminate this variable from all other equations by using (R1).
4) Iterate with the first occurring variable in the remaining equations.

II. Upwards

1) Let x_i be the first variable in the last equation. Eliminate x_i from all other equations by using (R1).
2) Go to previous equations and iterate.

Remark: The above algorithm always works, but in practice there are sometimes shorter / better ways to obtain the reduced echelon form.

Exercise 1. Which of the following linear systems are on row-reduced echelon form? For those that are not, find an equivalent system (i.e. one which has the same solutions) that is on row-reduced echelon form. For each system, find all solutions.

i) $\begin{cases} 3x_1 + 4x_2 + 7x_3 + 9x_4 = 5 \\ x_1 + x_2 - x_4 = 0 \end{cases}$

ii) $\{ x_1 + x_2 + x_3 = 1$

Exercise 2. Decide for which real numbers $a \in \mathbb{R}$ the following linear system has solutions. Give all the solutions in these cases.

$$\begin{cases} 2x_2 + 2x_3 = 2a + 8 \\ x_1 + x_2 - x_3 = 4 \\ 2x_1 - 4x_3 = a + 9 \end{cases}.$$