Tutorial 15: Orthogonal complement & Normal equation

Exercise 1. We define the subspace $U = \operatorname{span}\{u_1, u_2, u_3\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

(i) Determine a basis $B = (b_1, \dots, b_m)$ of U and calculate its dimension.

(ii) Determine a basis for $U^\perp$.

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map with matrix $A = [F] \in \mathbb{R}^{m \times n}$ and let $y \in \mathbb{R}^m$ be an arbitrary vector.

If $y \in \operatorname{im}(F)$ then the linear system $Ax = y$ has a solution. But if $y \notin \operatorname{im}(F)$ then there does not exists a $x \in \mathbb{R}^n$ with $Ax = y$. In this case, we can ask for the best possible x, i.e. the one such that $\|Ax - y\|$ is minimal.

Facts:

(i) The $x \in \mathbb{R}^n$ such that $\|Ax - y\|$ is minimal is given by a solution of the **normal equation**

$$A^T A x = A^T y.$$

(ii) The normal equation always has (at least one) solution x. This x has the property $Ax = P_{\operatorname{im}(F)}(y)$, i.e. Ax is the orthogonal projection of y onto the image of F.

(iii) If $\ker(A) = \{0\}$ (the columns of A are linearly independent) then $A^T A \in \mathbb{R}^{n \times n}$ is invertible and the normal equation has a unique solution given by

$$x = (A^T A)^{-1} A^T y.$$

Exercise 2. Assume we have the following data points

i	1	2	3
x_i	0	1	2
y_i	2	1	3

Find the line of best fit for the above data, i.e. find $a, b \in \mathbb{R}$ such that the function $l(x) = ax + b$ minimizes the sum of squares $\sum_{i=1}^{3} (l(x_i) - y_i)^2$.

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(i) We calculate the rref of $\begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix}$:

$$\begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that $B = (u_1, u_2)$ is a basis of U.

(Also notice $[u_3]_B = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$)

(ii) We want to find all $x \in \mathbb{R}^3$ such that $x \cdot u = 0$ for all $u \in U$.

We learned: Just need to check $x \cdot u_1 = x \cdot u_2 = 0$

basis of U

So we want x with $\begin{pmatrix} - u_1 - \\ - u_2 - \end{pmatrix} x = 0$

$$\rightsquigarrow \quad \begin{pmatrix} - u_1 - \\ - u_2 - \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$X = \begin{pmatrix} -t \\ -2t \\ t \end{pmatrix} \quad \text{for } t \in \mathbb{R}.$$

Therefore $U^\perp = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\}$ and $\left(\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right)$ is a basis of $U^\perp$.

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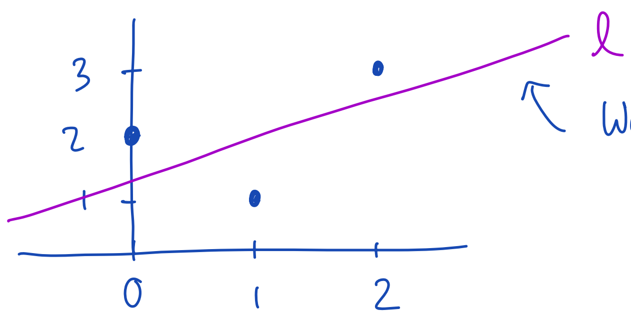

Want to find this.

If the 3 points would lie on a line $\ell(x) = ax+b$ would have $\ell(x_i) = ax_i + b = y_i$ for $i = 1,2,3$ and

$$\underbrace{\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \end{pmatrix}}_{A} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ax_1+b \\ ax_2+b \\ ax_3+b \end{pmatrix} = \begin{pmatrix} \ell(x_1) \\ \ell(x_2) \\ \ell(x_3) \end{pmatrix} = \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{y}$$

But they are not on a line and $A\begin{pmatrix}a\\b\end{pmatrix}=y$ has no solution.

Since $\left\|A\begin{pmatrix}a\\b\end{pmatrix}-y\right\| = \left\|\begin{pmatrix}\ell(x_1)\\\ell(x_2)\\\ell(x_3)\end{pmatrix}-\begin{pmatrix}y_1\\y_1\\y_2\end{pmatrix}\right\|$

$$= \sqrt{\sum_{i=1}^{3}\left(\ell(x_i)-y_i\right)^2}$$

we need to find $\begin{pmatrix}a\\b\end{pmatrix}$ such that $\left\|A\begin{pmatrix}a\\b\end{pmatrix}-y\right\|$ is minimal in order to minimize the sum of squares $\sum_{i=1}^{3}\left(\ell(x_i)-y_i\right)^2$.

$\Rightarrow$ Need to solve the normal equation

$$A^TA\begin{pmatrix}a\\b\end{pmatrix} = A^Ty$$

$$A^TA = \begin{pmatrix}0&1&2\\1&1&1\end{pmatrix}\begin{pmatrix}0&1\\1&1\\2&1\end{pmatrix} = \begin{pmatrix}5&3\\3&3\end{pmatrix}$$

$$A^Ty = \begin{pmatrix}0&1&2\\1&1&1\end{pmatrix}\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}7\\6\end{pmatrix}$$

$$\frac{1}{3} \quad (-1) \left(\begin{array}{cc|c} 5 & 3 & 7 \\ 3 & 3 & 6 \end{array}\right) \sim (-2) \left(\begin{array}{cc|c} 2 & 0 & 1 \\ 1 & 1 & 2 \end{array}\right)$$

$$\sim \left(-\frac{1}{2}\right) \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -3 \end{array}\right) \sim (-1) \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \end{array}\right)$$

$$\Rightarrow \quad \ell(x) = \frac{1}{2} x + \frac{3}{2}$$

Bonus question:

Consider the linear map

$$F : \mathbb{R}^2 \to \mathbb{R}^3$$
$$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} x .$$

i) Decide if $y = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ is in $\mathrm{im}(F)$.

ii) What is $P_{\mathrm{im}(F)}(y)$?

i&ii) By the previous exercise we know that

$$A\begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix} = P_{im(f)}(y)$$

$$\|$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ 2 \\ \frac{5}{2} \end{pmatrix}$$