Tutorial 14: Orthonormal basis & Gram-Schmidt

A basis $F = (f_1, \dots, f_m)$ of a subspace U is called an **orthonormal basis (ONB)** of U if $f_1, \dots, f_m$ are orthonormal. In other words, this means that:

(i) The vectors are pairwise orthogonal: For $1 \leq i \neq j \leq m$ we have $f_i \bullet f_j = 0$.

(ii) The vectors are normalized to norm 1: For all $1 \leq i \leq m$ we have $\|f_i\| = \sqrt{f_i \bullet f_i} = 1$.

Gram-Schmidt algorithm (GSA)

Let $B = (b_1, \dots, b_m)$ be an arbitrary basis of a subspace $U \subset \mathbb{R}^n$. The GSA constructs an orthonormal basis $F = (f_1, \dots, f_m)$ of U out of the basis B in the following m steps:

Step 1: Set $f_1 = \widehat{b_1} = \frac{1}{\|b_1\|} b_1$.

Step l ($2 \leq l \leq m$): We have constructed orthonormal vectors $f_1, \dots, f_{l-1}$ in the steps before. Now set

$$w_l = b_l - (b_l \bullet f_1) f_1 - \cdots - (b_l \bullet f_{l-1}) f_{l-1} = b_l - \sum_{i=1}^{l-1} (b_l \bullet f_i) f_i$$

and define $f_l = \frac{1}{\|w_l\|} w_l$.

Exercise 1. (Continue from the lecture) Consider the basis $B = (b_1, b_2, b_3)$ of $\mathbb{R}^3$, where

$$b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad b_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \qquad b_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

(i) Use the Gram-Schmidt algorithm to construct an orthonormal basis F of $\mathbb{R}^3$ out of B.

(ii) Find orthonormal bases for the subspaces $U = \operatorname{span}\{b_1, b_2\}$ and $V = \operatorname{span}\{b_1, b_3\}$.

See `https://tinyurl.com/yc5mjem6` for a visualization.