Tutorial 12: Basis & Coordinates (Preview)

Happy new year! あけましたおめでとうございます! Frohes neues Jahr!

Recall: Let $V \subset \mathbb{R}^n$ be a subspace. Vectors $v_1, \dots, v_l \in V$ form a **basis of V** if

(i) $V = \mathrm{span}\{v_1, \dots, v_l\}$,

(ii) $v_1, \dots, v_l$ are linearly independent.

In this case, we say that $(v_1, \dots, v_l)$ is a (ordered) basis of V.

(The following will be defined tomorrow in Lecture 12)
Let $B = (b_1, \dots, b_m)$ be a basis of a subspace $V \subset \mathbb{R}^n$. We define the **coordinate map** by

$$\begin{aligned} c_B : \mathbb{R}^m &\longrightarrow \mathbb{R}^n \\ \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} &\longmapsto \lambda_1 b_1 + \cdots + \lambda_m b_m \,. \end{aligned}$$

(i) Since $b_1, \dots, b_m$ are linearly independent the map c_B is injective.

(ii) Since $b_1, \dots, b_m$ span V, i.e. $V = \mathrm{span}\{b_1, \dots, b_m\}$, we have $\mathrm{im}(c_B) = V$.

(iii) From (i) and (ii) we get: The map $c_B : \mathbb{R}^m \longrightarrow V$ is bijective.

(iv) For all $x \in V$ there exist unique $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that

$$x = \lambda_1 b_1 + \cdots + \lambda_m b_m \,.$$

The numbers $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ are the **coordinates of x (in the basis B)**.

(v) The **coordinate vector** of x (with respect to B) is given by

$$[x]_B = c_B^{-1}(x) = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} .$$

Exercise 1. Consider the subspace $V = \mathrm{span}\{v_1, v_2\}$ of $\mathbb{R}^3$ where

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

(i) Find two different bases B_1 and B_2 of V.

(ii) For the element $x = \begin{pmatrix} 5 \\ -2 \\ -8 \end{pmatrix}$ decide if x is in V. If this is the case, calculate the coordinate vectors $[x]_{B_1}$ and $[x]_{B_2}$.