Tutorial 11: Linear independence & Bases

(i) Vectors $v_1, \dots, v_l \in \mathbb{R}^n$ are called **linearly independent** if the equation

$$\lambda_1 v_1 + \dots + \lambda_l v_l = 0 \tag{0.1}$$

with $\lambda_1, \dots, \lambda_l \in \mathbb{R}$ just has the unique solution $\lambda_1 = \dots = \lambda_l = 0$.

(ii) If there exist another solution of (0.1), i.e. where at least for one $j = 1, \dots, l$ we have $\lambda_j \neq 0$, then the vectors $v_1, \dots, v_l$ are called **linearly dependent**.

Let $V \subset \mathbb{R}^n$ be a subspace. Vectors $v_1, \dots, v_l \in V$ form a **basis of V** if

(i) $V = \operatorname{span}\{v_1, \dots, v_l\}$,

(ii) $v_1, \dots, v_l$ are linearly independent.

In this case, we say that $\{v_1, \dots, v_l\}$ is a basis of V.

Exercise 1. Consider the following linear map

$$\begin{aligned} F : \mathbb{R}^5 &\longrightarrow \mathbb{R}^3 \\ x &\longmapsto \begin{pmatrix} -2 & 2 & 2 & 0 & 6 \\ -2 & 2 & 1 & -3 & 5 \\ -3 & 3 & 2 & -3 & 8 \end{pmatrix} x\,. \end{aligned}$$

(i) Find a basis for $\ker(F)$.

(ii) Find a basis for $\operatorname{im}(F)$.

Plan for the coming weeks:

(i) Friday 22nd December during the lecture: Christmath Challenge 2023 (45minutes) & Lecture 11 (45 minutes). Make sure to be on time and bring your phone or laptop to take the challenge (we will again use menti.com). Content of the challenge: Lecture 1 - 10 (and more...)

(ii) Tuesday 26th December: No tutorials (Also no Calculus tutorial)

(iii) First meeting next year: Tuesday 9th January for the tutorial.

(iv) **Wednesday 10th January**: Lecture 12. This is a makeup day for Friday 12th January, where we will have no lecture.

Homework 6: Linear independence & Basis

Deadline: 14th January, 2024

Exercise 1. (6 Points) Let $V \subset \mathbb{R}^n$ be a subspace, $v_1, \dots, v_l \in V$ linearly independent and $V = \operatorname{span}\{w_1, \dots, w_m\}$ for some $w_1, \dots, w_m \in \mathbb{R}^n$. Show that we have $l \leq m$. (Without using Lemma 9.4)

In other words: Show that a subspace spanned by m vectors can not contain more than m linearly independent vectors.

Exercise 2. (7 Points) Determine bases for the kernel and the image of the following linear map

$$\begin{aligned} F : \mathbb{R}^5 &\longrightarrow \mathbb{R}^3 \\ x &\longmapsto \begin{pmatrix} 0 & 0 & 0 & -2 & 2 \\ -1 & -2 & 1 & 1 & 2 \\ 1 & 2 & -1 & 2 & -5 \end{pmatrix} x \,. \end{aligned}$$

The following exercise is intended to show the basic idea of 3D computer graphics, by showing how to get a 2-dimensional picture (to be shown on a 2-dimensional monitor) from an 3-dimensional object.

Exercise 3. (7 Points)

(i) We define the corners of a cube with side length 18 in $\mathbb{R}^3$ by the following set of 8 points:

$$W = \left\{ \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3 \mid w_1, w_2, w_3 \in \{0, 18\} \right\} .$$

Make a drawing of a cube with side length 18 in $\mathbb{R}^3$, i.e. draw the 8 points in the set W and connect two points if they differ just by one entry.

(This just means that you draw a cube like you would usually draw it. ”Differ by one entry” just means that these points are on the same edge of the cube.)

(ii) Show that $D = (d_1, d_2, d_3)$ is a basis of $\mathbb{R}^3$, where

$$d_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad d_2 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \quad d_3 = \begin{pmatrix} 3 \\ -6 \\ 3 \end{pmatrix} .$$

(iii) Write each $x \in W$ as a linear combination in the basis D, i.e. for each x find $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ with

$$x = \lambda_1 d_1 + \lambda_2 d_2 + \lambda_3 d_3 \,.$$

(iv) For each $x \in W$ draw the points (λ_1, λ_2) in $\mathbb{R}^2$. Connect two points if the corresponding elements in W just differ by one entry.

Explanation: What you should get in (iv) is a drawing of the 3-dimensional cube in 2 dimensions. The basis D somehow describes from which direction you look at the cube. If you replaced the D by the standard basis (e_1, e_2, e_3), you would get a picture of the cube from the top (i.e., just a square). The λ_3, which you did not use for the drawing, describes the distance in the viewing direction.

Exercise 1. Consider the following linear map

$$F : \mathbb{R}^5 \longrightarrow \mathbb{R}^3$$
$$x \longmapsto \begin{pmatrix} -2 & 2 & 2 & 0 & 6 \\ -2 & 2 & 1 & -3 & 5 \\ -3 & 3 & 2 & -3 & 8 \end{pmatrix} x\,.$$

(i) Find a basis for $\ker(F)$.

(ii) Find a basis for $\operatorname{im}(F)$.

(i) We calculate the Kernel by solving $F(x) = 0$, i.e. we determine $\mathrm{rref}([F])$:

$$[F] = \begin{array}{c} -\tfrac{1}{2} \;\; -1 \\ \\ \end{array} \begin{pmatrix} -2 & 2 & 2 & 0 & 6 \\ -2 & 2 & 1 & -3 & 5 \\ -3 & 3 & 2 & -3 & 8 \end{pmatrix} \sim \begin{array}{c} 3 \\ -1 \\ \end{array} \begin{pmatrix} 1 & -1 & -1 & 0 & -3 \\ 0 & 0 & -1 & -3 & -1 \\ -3 & 3 & 2 & -3 & 8 \end{pmatrix}$$

$$\sim \begin{array}{c} \\ 1 \\ \end{array} \begin{pmatrix} 1 & -1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 3 & -2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{rref}([F])$$

$\Rightarrow$ The solutions of $F(x) = 0$ are given by $x = \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix}$

with
$$\begin{aligned} x_1 &= t_1 - 3t_2 + 2t_3 \\ x_2 &= t_1 \\ x_3 &= -3t_2 - t_3 \\ x_4 &= t_2 \\ x_5 &= t_3 \end{aligned} \qquad \text{for } t_1, t_2, t_3 \in \mathbb{R}$$

Written differently:

$$X = t_1 \cdot \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{v_1} + t_2 \underbrace{\begin{pmatrix} -3 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}}_{v_2} + t_3 \underbrace{\begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}}_{v_3} \quad (*)$$

$\Rightarrow \quad \ker(F) = \text{span}\{v_1, v_2, v_3\}$

Notice that v_1, v_2, v_3 are lin. indep. which follows by looking at rows 2, 4, 5.

$\Rightarrow \quad \{v_1, v_2, v_3\}$ is basis of $\ker(F)$.

(ii) Image: Write $[F] = \begin{pmatrix} | & | & | & | & | \\ u_1 & u_2 & u_3 & u_4 & u_5 \\ | & | & | & | & | \end{pmatrix}$.

We have $\text{im}(F) = \text{span}\{u_1, u_2, u_3, u_4, u_5\}$, but $u_1, \dots, u_5$ are not lin. indep. as we have seen in (i). Since any $\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_5 \end{pmatrix} \in \ker(F)$ gives $\lambda_1 u_1 + \dots + \lambda_5 u_5 = 0$.

Claim 1: $u_2, u_4, u_5 \in \text{span}\{u_1, u_3\}$

Setting $t_1 = 1, t_2 = 0, t_3 = 0$ in ($\star$)

gives $u_1 + u_2 = 0 \Rightarrow u_2 = -u_1 \in \text{span}\{u_1, u_3\}$

$t_1 = 0, t_2 = 1, t_3 = 0$ in ($\star$) gives

$-3u_1 - 3u_3 + u_4 = 0 \Rightarrow u_4 = 3u_1 + 3u_3 \in \text{span}\{u_1, u_3\}$

$t_1 = 0, t_2 = 0, t_3 = 1$ in ($\star$) gives

$2u_1 - u_3 + u_5 = 0 \Rightarrow u_5 = -2u_1 + u_3 \in \text{span}\{u_1, u_3\}$

Claim 2: u_1, u_3 are lin. indep.

Need to check $\lambda_1 u_1 + \lambda_3 u_3 = 0 \Rightarrow \lambda_1 = \lambda_3 = 0$.

But by (i) we see $\begin{pmatrix} | & | \\ u_1 & u_3 \\ | & | \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

i.e. u_1, u_3 are lin. indep. $\Rightarrow \{u_1, u_3\}$ basis of $\text{im}(F)$