Tutorial 10

Homework 5: Subspaces

Deadline: 17th December, 2023

Exercise 1. (5+3 = 8 Points)

(i) Which of the following subsets are subspaces? Justify your answers.

$$U_1 = \left\{ x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = x_3^3 \right\},$$
$$U_2 = \left\{ x \in \mathbb{R}^4 \mid x_1 - x_2 = 2x_3 \text{ and } x_2 + x_3 = -x_4 \right\},$$
$$U_3 = \{ x \in \mathbb{R}^n \mid Ax = x \}, \quad \text{where } A \in \mathbb{R}^{n\times n} \text{ is a fixed matrix},$$
$$U_4 = \{ x \in \mathbb{R}^n \mid x \bullet v = 0 \}, \quad \text{for a fixed } v \in \mathbb{R}^n,$$
$$U_5 = \left\{ x \in \mathbb{R}^2 \mid x_1 \le x_2 \right\}.$$

(ii) For each subset U in (i) that is a subspace, find numbers $a, b \geq 1$ and a linear map $F : \mathbb{R}^a \to \mathbb{R}^b$ such that $\ker(F) = U$.

Exercise 2. (2+3+3 = 8 Points) Consider the following subspace

$$W = \ker(P_u) = \left\{ x \in \mathbb{R}^3 \mid P_u(x) = 0 \right\}, \quad \text{where } u = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

(i) Determine vectors $v_1, \ldots, v_m \in \mathbb{R}^3$ with $W = \operatorname{span}\{v_1, \ldots, v_m\}$.

(ii) Find a linear map $H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with $\operatorname{im}(H) = W$.

(iii) Calculate $\ker(H)$ and $\ker(P_u \circ H)$.

Exercise 3. (3+3 = 6 Points)

(i) Let $U, V \subset \mathbb{R}^m$ be subspaces. Decide whether the union $U \cup V$ is also a subspace or not.

(ii) Let $U, V \subset \mathbb{R}^m$ be subspaces. Decide whether the intersection $U \cap V$ is also a subspace or not.

Recall

A subset $U \subset \mathbb{R}^n$ is a **subspace** of $\mathbb{R}^n$ if

1. (i) $0 \in U$,
2. (ii) for all $u, v \in U$ we have $u + v \in U$,
3. (iii) for all $u \in U$ and $\lambda \in \mathbb{R}$ we have $\lambda u \in U$.

Example Exercire from Final 2020

Set $u = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Which of the following subsets of $\mathbb{R}^2$ are subspaces? Justify your answers.

i) $U_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid 2x_1 - 3x_2 = x_1 \right\}$. $\iff$ $x_1 - 3x_2 = 0$

ii) $U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 \text{ is an integer, i.e. } x_1 \in \{\ldots, -2, -1, 0, 1, 2, \ldots\} \right\}$.

iii) $U_3 = \{x \in \mathbb{R}^2 \mid x \notin \operatorname{span}\{u\}\}$.

iv) $U_4 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \bullet u = x_1 \right\}$.

i) ① We have $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in U_1$ since $2 \cdot 0 - 3 \cdot 0 = 0$.

② If $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in U_1$ then $u_1 - 3u_2 = 0$.
$v_1 - 3v_2 = 0$

Then $u + v = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$ satisfies

$$(u_1 + v_1) - 3(u_2 + v_2)$$

$$= u_1 - 3u_2 + v_1 - 3v_2 = 0 + 0 = 0$$

$$\Rightarrow u + v \in U_1$$

③ If $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in U_1$ and $\lambda \in \mathbb{R}$ then

$\lambda u = \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \end{pmatrix}$ satisfies $\lambda u_1 - 3\lambda u_2 = \lambda \underbrace{(u_1 - 3u_2)}_{=0} = 0$

(Other solution check Final 2020 Solutions) $\Rightarrow \lambda u \in U_1$

Example for Kernel:

Consider the linear map

$$G: \mathbb{R}^4 \longrightarrow \mathbb{R}^2$$
$$x \longmapsto \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} x$$

Kernel: Solve $[G]x = 0$

$$\left([G] \,\middle|\, \begin{matrix} 0 \\ 0 \end{matrix} \right) = \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array} \right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array} \right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array} \right)$$

Solution:
$$x_1 = -2t_1 - t_2$$
$$x_2 = t_1$$
free var.
$$x_3 = t_1$$
$$x_4 = t_2$$

Another way of writing:

$$x = t_1 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore:

$$\ker(G) = \left\{ t_1 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; t_1, t_2 \in \mathbb{R} \right\}.$$

$$= \operatorname{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Check yourself: $\operatorname{im}(G) = \mathbb{R}^2$

Recall:

The span of $v_1, \dots, v_n \in \mathbb{R}^m$ is the set of all linear combinations:

$$\operatorname{span}\{v_1, \dots, v_n\} = \{ \lambda_1 v_1 + \dots + \lambda_n v_n \in \mathbb{R}^m \mid \lambda_1, \dots, \lambda_n \in \mathbb{R} \}.$$

Notice: If $F: \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto Ax$ with $A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$

then $\operatorname{im}(F) = \operatorname{span}\{v_1, \dots, v_n\}$.