Linear Algebra I - Midterm question session

15th November 2023, 20:00

- You can join/leave whenever you want
- You can ask questions via voice, chat or via menti
- www.menti.com: 42 66 05 2
- I would appreciate if you turn on your cameras

1) (10 Points) Consider the following linear system

$$\left\{\begin{array}{ccccccccc} 2x_1 & + & 3x_2 & + & 4x_3 & + & 5x_4 & = & 6 \\ x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & 5 \\ 3x_1 & + & 4x_2 & + & 5x_3 & + & 6x_4 & = & 7 \end{array}\right. .$$

i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

$(A|b) \sim \dots \sim \underbrace{(B|c)}_{\mathrm{rref}(A|b)}$

$B = \mathrm{rref}(A)$

ii) Determine the row-reduced echelon forms of the matrices $(A \mid b)$ and A.

iii) Find all the solutions to the linear system.

iv) Calculate the rank of $(A \mid b)$ and A.

v) Find a vector $c \in \mathbb{R}^3$, such that $Ax = c$ has no solutions. Calculate the rank of $(A \mid c)$.

2) (10 Points) Let $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in \mathbb{R}^2$ and define the following four functions:

$$f_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad x \longmapsto (u \bullet x)u + x\,,$$

$$f_2 : \mathbb{R} \longrightarrow \mathbb{R}^2, \quad x \longmapsto \begin{pmatrix} 2\cos(x) \\ \sin(x) \end{pmatrix}\,,$$

$$f_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad x \longmapsto \frac{x \bullet x}{u \bullet u} u\,,$$

$$f_4 : \mathbb{R}^3 \longrightarrow \mathbb{R}^4, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ x_1 + 2x_2 + 3x_3 \\ x_1 - x_3 \\ x_2 \end{pmatrix}.$$

i) Which of the above functions f_1, f_2, f_3, f_4 are linear maps? For each one that is linear, determine its matrix.

ii) Draw a picture of the image of f_2. Is f_2 injective and/or surjective?

3) (6 Points) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad G\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

i) Determine the matrix of G.

ii) Determine the matrix of $G \circ G$.

4) (6 Points) We define the following linear map

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_2 + x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}.$$

i) Calculate the image of H.

ii) Decide if H is injective and/or surjective.

iii) Find all vectors $v \in \mathbb{R}^3$, which are orthogonal to all vectors in the image of H.

1) (10 Points) Consider the following linear system

$$\begin{cases} -2x_1 + 4x_2 + x_3 + x_4 = 6 \\ -3x_1 + 6x_2 + x_3 = 7 \\ x_1 - 2x_2 + x_4 = -1 \end{cases}.$$

i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

ii) Determine the row-reduced echelon forms of the matrices $(A \mid b)$ and A.

iii) Find all the solutions to the linear system.

iv) Calculate the rank of $(A \mid b)$ and A.

v) Find all $y \in \mathbb{R}^4$ with $Ay = 2b$ by using your result for iii).

2) (8 Points) Let $u = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \in \mathbb{R}^2$ and define the following four functions:

$x = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\lambda = 2$

$$f_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \quad x \longmapsto \begin{pmatrix} u \bullet x \\ 0 \\ x \bullet u \end{pmatrix}, \qquad f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto 2^{x_1+x_2} - 1, \qquad f_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 - 3x_2 \\ 2x_1 + x_2x_3 \end{pmatrix}.$$

$f_3(\lambda x)$

$\lambda f_3(x)$

i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.

ii) Is f_2 injective and/or surjective?

3) (8 Points)

i) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad G\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Determine the matrix of G.

ii) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a function with

$$F\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad F\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad F\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

Show that F is <u>not</u> a linear map.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_1 - x_3 \\ x_2 + x_3 \end{pmatrix}.$$

i) Calculate the image of H.

ii) Decide if H is injective and/or surjective.

iii) Find a non-zero vector $v \in \mathbb{R}$, such that v is orthogonal to $H(v)$. (Just <u>one explicit</u> vector is enough)

After finishing this exam please send your solution as one pdf file to
henrik.bachmann@math.nagoya-u.ac.jp

1) (10 Points) Consider the following linear system

$$\left\{\begin{array}{rcrcrcrcr} 3x_1 & - & 6x_2 & + & x_3 & + & 5x_4 & = & 5 \\ 2x_1 & - & 4x_2 & + & x_3 & + & 3x_4 & = & 4 \\ -x_1 & + & 2x_2 & - & 2x_3 & & & = & -5 \end{array}\right. .$$

i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

ii) Determine the row-reduced echelon forms of the matrices $(A \mid b)$ and A and calculate their ranks.

iii) Find all the solutions to the linear system.

iv) Determine all $x \in \mathbb{R}^4$ which satisfy $Ax = b$ and which are orthogonal to the vector $u = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$.

2) (8 Points) Let $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$ and define the following three functions:

$$f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_1 + 3x_2 \\ x_1 + (u \bullet u)x_3 \end{pmatrix},$$

$$f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \sin(x_1) + \cos(x_2),$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \quad x \longmapsto \begin{pmatrix} x \bullet x \\ 0 \\ u \bullet u \end{pmatrix}.$$

i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.

ii) Is f_2 injective and/or surjective?

3) (8 Points)

i) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad G\begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

Determine the matrix of G.

ii) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map with

$$F\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, \quad F\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 6 \end{pmatrix}.$$

Show that F is not injective.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 - x_3 \\ x_1 + 2x_2 \\ x_2 + x_3 \end{pmatrix}.$$

i) Calculate the image of H.

ii) Decide if H is injective and/or surjective.

iii) Find all vectors $x \in \mathbb{R}^3$ with $H(x) = 2x$.

After finishing this exam submit your solution as one pdf file at NUCT at the "Midterm" assignment.

Linear Algebra I - Midterm exam Fall 2022
Nagoya University, G30 Program Instructor: Henrik Bachmann

---

1) (10 Points) Consider the following linear system

$$\begin{cases} x_1 + 3x_2 + x_4 = 1 \\ x_2 + 2x_3 - 2x_4 = 2 \\ 2x_1 - 2x_2 + x_3 + x_4 = 3 \end{cases}.$$

i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

ii) Determine the row-reduced echelon forms of the matrices $(A \mid b)$ and A and calculate their ranks.

iii) Find all the solutions to the linear system.

iv) Determine all $x \in \mathbb{R}^4$ which satisfy $Ax = b$ and which have norm $\|x\| = \sqrt{14}$.

2) (8 Points) Let $u = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \mathbb{R}^2$ and define the following three functions:

$-10 = 10 - 20$

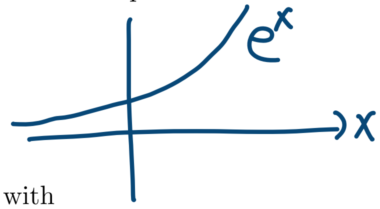
$$f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} (u \bullet u) - 2 \\ x_1 + (u \bullet u)x_3 \end{pmatrix},$$

$$f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto e^{x_1} - e^{x_2},$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad x \longmapsto (x \bullet u)u.$$

i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.

ii) Is f_2 injective and/or surjective?



3) (8 Points) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}, \quad G\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \end{pmatrix}.$$

i) Determine the matrix of G.

ii) Find all vectors $x \in \mathbb{R}^2$ such that x is orthogonal to $G(x)$.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^4 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_1 + 2x_2 + x_3 \\ 2x_1 - 2x_3 \end{pmatrix}.$$

i) Calculate the image of H.

ii) Decide if H is injective and/or surjective.

iii) Find a linear map $F : \mathbb{R}^2 \to \mathbb{R}^4$ with $\mathrm{im}(F) = \mathrm{im}(H)$.

2022

3) (8 Points) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}4\\-2\end{pmatrix}, \quad G\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}-4\\4\end{pmatrix}.$$

i) Determine the matrix of G.

ii) Find all vectors $x \in \mathbb{R}^2$ such that x is orthogonal to $G(x)$.

i) Want $[G]$

One way: Find $G\begin{pmatrix}x_1\\x_2\end{pmatrix}$. For this we try to find a, b with $\begin{pmatrix}x_1\\x_2\end{pmatrix} = a\begin{pmatrix}-1\\1\end{pmatrix} + b\begin{pmatrix}2\\2\end{pmatrix}$

$$= \begin{pmatrix}-1 & 2\\1 & 2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$$

$$\left(\begin{array}{cc|c}-1 & 2 & x_1\\1 & 2 & x_2\end{array}\right) \sim \left(\begin{array}{cc|c}-1 & 2 & x_1\\0 & 4 & x_1+x_2\end{array}\right)$$

$$\sim \left(\begin{array}{cc|c}1 & -2 & -x_1\\0 & 1 & \frac{x_1}{4}+\frac{x_2}{4}\end{array}\right)$$

$$\sim \left(\begin{array}{cc|c}1 & 0 & -\frac{x_1}{2}+\frac{x_2}{2}\\0 & 1 & \frac{x_1}{4}+\frac{x_2}{4}\end{array}\right) \quad \begin{matrix}= a\\= b\end{matrix}$$

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = \left(-\frac{x_1}{2}+\frac{x_2}{2}\right)\begin{pmatrix}-1\\1\end{pmatrix} + \left(\frac{x_1}{4}+\frac{x_2}{4}\right)\begin{pmatrix}2\\2\end{pmatrix}$$

$$\Rightarrow G\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(-\frac{x_1}{2} + \frac{x_2}{2}\right)\begin{pmatrix} 4 \\ -2 \end{pmatrix} + \left(\frac{x_1}{4} + \frac{x_2}{4}\right)\begin{pmatrix} -4 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -3x_1 + x_2 \\ 2x_1 \end{pmatrix} = \underbrace{\begin{pmatrix} -3 & 1 \\ 2 & 0 \end{pmatrix}}_{[G]}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Another way:

Try to find $G\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $G\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

since $[G] = \left(G\begin{pmatrix} 1 \\ 0 \end{pmatrix} \; G\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} -3 & 1 \\ 2 & 0 \end{pmatrix}$

$$\left(G(e_1) \; G(e_2) \right)$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$G\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} G\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{4} G\begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 4 \\ -2 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} -4 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} 4 \\ -2 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} -4 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$F : \mathbb{R}^3 \to \mathbb{R}^3$$

$$[F] = \begin{pmatrix} 1 & 0 & 9 \\ 2 & 4 & 10 \\ 3 & 8 & 0 \end{pmatrix}$$

$$F\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = [F]\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$F\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = [F] \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 8 \end{pmatrix}$$

---

ii) Want to find $\underset{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}{\underset{\shortparallel}{x}} \in \mathbb{R}^2$ with $\underbrace{x \bullet G(x) = 0}$

x orthogonal to G(x).

$$x \bullet G(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \bullet G\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\overset{i)}{=} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \bullet \begin{pmatrix} -3x_1 + x_2 \\ 2x_1 \end{pmatrix}$$

$$= x_1 \cdot (-3x_1 + x_2) + 2x_1 x_2$$

$$= x_1 \cdot (\underbrace{-3x_1 + x_2 + 2x_2}_{-3x_1 + 3x_2}) = 0$$

We have $x \bullet G(x) = 0$ if $x_1 = 0$

or $-3x_1 + 3x_2 = 0$.

$(-x_1 + x_2 = 0)$

(In other words if $x = \begin{pmatrix} 0 \\ t \end{pmatrix}$ or $x = \begin{pmatrix} t \\ t \end{pmatrix}$ for $t \in \mathbb{R}$.)

$$f: \mathbb{R}^2 \to \mathbb{R}$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1 + \sin(x_2)$$

$\{y \in \mathbb{R} \mid \exists x \in \mathbb{R}^2$ with $f(x) = y\}$ $=$ $im(f) = \mathbb{R}$

Q: 1) Is f surjective?

Yes: For any $y \in \mathbb{R}$ we can choose $x = \begin{pmatrix} y \\ 0 \end{pmatrix}$ and have $f(x) = y$. Therefore any y is in $im(f)$. $\Rightarrow im(f) = \mathbb{R}$.

2) Is f injective?

No since $f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = f\begin{pmatrix} 1 \\ 2\pi \end{pmatrix}$.

$$g: \mathbb{R}^2 \to \mathbb{R}$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1^2 + \sin(x_2)$$

g is not surjective since $-5 \notin im(g)$.

(Because $x_1^2 \geq 0$ for any $x_1 \in \mathbb{R}$

and $\sin(x_2) \geq -1$ for any $x_2 \in \mathbb{R}$.

$\Rightarrow$ $x_1^2 + \sin(x_2) \geq -1$)

2019

3) (6 Points) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}, \quad G\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}.$$

i) Determine the matrix of G.

ii) Determine the matrix of $G \circ G$.

i) $$[G] = \begin{pmatrix}-5 & 4\\-2 & 3\end{pmatrix}$$

$$G\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}-5x_1 + 4x_2\\-2x_1 + 3x_2\end{pmatrix}$$

ii) $$(G \circ G)\begin{pmatrix}x_1\\x_2\end{pmatrix} = G\left(G\begin{pmatrix}x_1\\x_2\end{pmatrix}\right)$$

$$= G\begin{pmatrix} -5x_1 + 4x_2 \\ -2x_1 + 3x_2 \end{pmatrix}$$

$$= \begin{pmatrix} -5(-5x_1+4x_2) + 4(-2x_1+3x_2) \\ -2(-5x_1+4x_2) + 3(-2x_1+3x_2) \end{pmatrix}$$

$$= \begin{pmatrix} 17x_1 - 8x_2 \\ 4x_1 + x_2 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 17 & -8 \\ 4 & 1 \end{pmatrix}}_{[G \circ G] = [G] \cdot [G]} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2020:

$$f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto 2^{x_1+x_2} - 1,$$

$a^x > 0 \quad a \in \mathbb{R}$

f_2 surjective? No, since $2^x > 0$ for any $x \in \mathbb{R}$, i.e. $2^{x_1+x_2} - 1 > -1$ for any $x_1, x_2 \in \mathbb{R}$.

$\Rightarrow -100 \notin im(f_2)$

$\Rightarrow im(f_2) \neq \mathbb{R}$

f_2 injective? No, since $f_2\begin{pmatrix} 0 \\ 0 \end{pmatrix} = f_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$\| \\ 0$

//

If $F: \mathbb{R}^n \to \mathbb{R}^m$ is linear then

$$F(0) = 0$$

$$F\left(\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$