1) (10 Points) Consider the following linear system

$$\begin{cases} -x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 & = & 3 \\ 3x_1 & - & 2x_2 & - & x_3 & + & 2x_4 & = & 3 \\ x_1 & + & 2x_2 & + & 5x_3 & + & 6x_4 & = & 9 \end{cases}.$$

(i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

(ii) Calculate the row-reduced echelon forms of the matrices $(A \mid b)$ and A and calculate their ranks.

(iii) Determine all the solutions to the linear system $Ax = b$.

(iv) Find an injective linear map $F : \mathbb{R}^2 \to \mathbb{R}^4$ such that $Ax = 0$ for any $x \in \operatorname{im}(F)$.

2) (8 Points) Let $u = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \in \mathbb{R}^2$ and define the following three functions:

$$f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1(u \bullet u) - x_2 \\ x_1 + x_3 \end{pmatrix},$$

$$f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto x_1 \sin(x_2),$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad x \longmapsto (x \bullet u)x.$$

(i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.

(ii) Is f_2 injective and/or surjective?

3) (8 Points) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \quad G\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

(i) Determine the matrix of G.

(ii) Find all vectors $x \in \mathbb{R}^2$ such that x is orthogonal to every vector $v \in \operatorname{im}(G)$.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^4 \longrightarrow \mathbb{R}^3 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{pmatrix}.$$

(i) Calculate the image of H.

(ii) Decide if H is injective and/or surjective.

(iii) Find a linear map $J : \mathbb{R}^3 \to \mathbb{R}^4$ with $H(J(y)) = y$ for all $y \in \mathbb{R}^3$.

(iv) Show that there cannot exist a linear map $K : \mathbb{R}^3 \to \mathbb{R}^4$ with $K(H(x)) = x$ for all $x \in \mathbb{R}^4$.

1) (10 Points) Consider the following linear system

$$\left\{\begin{array}{rcrcrcrcl} -x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 & = & 3 \\ 3x_1 & - & 2x_2 & - & x_3 & + & 2x_4 & = & 3 \\ x_1 & + & 2x_2 & + & 5x_3 & + & 6x_4 & = & 9 \end{array}\right. .$$

(i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

(ii) Calculate the row-reduced echelon forms of the matrices $(A \mid b)$ and A and calculate their ranks.

(iii) Determine all the solutions to the linear system $Ax = b$.

(iv) Find an injective linear map $F : \mathbb{R}^2 \to \mathbb{R}^4$ such that $Ax = 0$ for any $x \in \operatorname{im}(F)$.

(i) $A = \begin{pmatrix} -1 & 2 & 3 & 2 \\ 3 & -2 & -1 & 2 \\ 1 & 2 & 5 & 6 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 3 \\ 9 \end{pmatrix}$

(ii) $(A|b) = \overset{(-1)\,(1)\,(3)}{} \left(\begin{array}{cccc|c} -1 & 2 & 3 & 2 & 3 \\ 3 & -2 & -1 & 2 & 3 \\ 1 & 2 & 5 & 6 & 9 \end{array}\right) \sim \overset{(\frac{1}{4})\,(-1)}{} \left(\begin{array}{cccc|c} 1 & -2 & -3 & -2 & -3 \\ 0 & 4 & 8 & 8 & 12 \\ 0 & 4 & 8 & 8 & 12 \end{array}\right)$

$\sim \overset{(2)}{} \left(\begin{array}{cccc|c} 1 & -2 & -3 & -2 & -3 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) = \operatorname{rref}(A|b)$

We get $\operatorname{rref}(A) = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\operatorname{rk}(A) = \operatorname{rk}(A|b) = 2$.

(iii) By (ii) the solutions to $Ax=b$ are given by $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

with
$$\begin{aligned} x_1 &= 3 - t_1 - 2t_2 \quad \text{for } t_1, t_2 \in \mathbb{R}. \\ x_2 &= 3 - 2t_1 - 2t_2 \\ x_3 &= t_1 \\ x_4 &= t_2 \end{aligned}$$

(iv) By (ii) the solutions to $Ax=0$ are $x=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

with
$$\begin{aligned} x_1 &= -t_1 - 2t_2 \qquad t_1, t_2 \in \mathbb{R}. \\ x_2 &= -2t_1 - 2t_2 \\ x_3 &= t_1 \\ x_4 &= t_2 \end{aligned}$$

Therefore we define

$$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^4$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} -x_1 - 2x_2 \\ -2x_1 - 2x_2 \\ x_1 \\ x_2 \end{pmatrix}.$$

Clearly $AF(x) = 0$.

F is injective, since if $F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = F\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$ we directly get $x_1 = \tilde{x}_1$ and $x_2 = \tilde{x}_2$ by comparing the third and fourth entry.

2) (8 Points) Let $u = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \in \mathbb{R}^2$ and define the following three functions:

$$f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1(u \bullet u) - x_2 \\ x_1 + x_3 \end{pmatrix},$$

$$f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto x_1 \sin(x_2),$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad x \longmapsto (x \bullet u)x.$$

(i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.

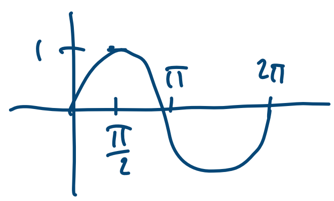
(ii) Is f_2 injective and/or surjective?

(i) $\underline{f_1}$: $u \bullet u = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2.2 + (-1)(-1) = 5$

$$\Rightarrow f_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5x_1 - x_2 \\ x_1 + x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 5 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{[f_1]} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\Rightarrow f_1$ is linear.

$\underline{f_2}$: For $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\pi}{2} \end{pmatrix}$



we have $f_2(x) = 1 \cdot \sin(\frac{\pi}{2}) = 1 \cdot 1 = 1$.

For $\lambda = 2$ we get $\lambda f_2(x) = 2$, but

$$f_2(\lambda x) = f_2\left(2 \cdot \begin{pmatrix} 1 \\ \frac{\pi}{2} \end{pmatrix}\right) = f_2 \begin{pmatrix} 2 \\ \pi \end{pmatrix} = 2 \cdot \sin(\pi) = 0.$$

$\Rightarrow \lambda f_2(x) \neq f_2(\lambda x)$

$\Rightarrow f_2$ is not linear.

(iii) $\underline{f_3}$:

$$f_3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (2x_1 - x_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1^2 - x_1 x_2 \\ 2x_1 x_2 - x_2^2 \end{pmatrix}.$$

We get

$$2 f_3\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\neq$$

$$f_3\left(2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = f_3\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

$\Rightarrow$ f_3 is not linear.

(ii)

- For any $y \in \mathbb{R}$ we can choose $x = \begin{pmatrix} y \\ \frac{\pi}{2} \end{pmatrix}$ and get $f_2(x) = f_2\begin{pmatrix} y \\ \frac{\pi}{2} \end{pmatrix} = y \cdot \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} = y$.

  Therefore $y \in \operatorname{im}(f_2)$ for any $y \in \mathbb{R}$ and f_2 is surjective.

- Since $f_2\begin{pmatrix} 1 \\ \pi \end{pmatrix} = f_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$ f_2 is not injective.

3) (8 Points) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \quad G\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

(i) Determine the matrix of G.

(ii) Find all vectors $x \in \mathbb{R}^2$ such that x is orthogonal to every vector $v \in \operatorname{im}(G)$.

(i) We try to find a, b with

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a\begin{pmatrix} 2 \\ -2 \end{pmatrix} + b\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

$$\frac{1}{2} \quad 1 \quad \left(\begin{array}{cc|c} 2 & -1 & x_1 \\ -2 & 2 & x_2 \end{array}\right) \sim \frac{1}{2} \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & \frac{x_1}{2} \\ 0 & 1 & x_1+x_2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & x_1+\frac{x_2}{2} \\ 0 & 1 & x_1+x_2 \end{array}\right)$$

$$\Rightarrow \quad a = x_1 + \frac{x_2}{2}, \quad b = x_1 + x_2.$$

Therefore

$$G\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = G\left(\left(x_1 + \frac{x_2}{2}\right)\begin{pmatrix} 2 \\ -2 \end{pmatrix} + (x_1+x_2)\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right)$$

$$\overset{G \text{ linear}}{=} \left(x_1 + \frac{x_2}{2}\right) G\begin{pmatrix} 2 \\ -2 \end{pmatrix} + (x_1+x_2)\, G\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$= \left(x_1 + \frac{x_2}{2}\right)\begin{pmatrix} 2 \\ -2 \end{pmatrix} + (x_1+x_2)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}}^{[G]}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(ii) By (i) we see that any $v \in \operatorname{im}(G)$ has the form $v = \begin{pmatrix} t \\ -t \end{pmatrix}$ for $t \in \mathbb{R}$.

We have

$$x \cdot v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} t \\ -t \end{pmatrix} = x_1 t - x_2 t$$

$$= (x_1 - x_2) t.$$

In order to get $x \cdot v = 0$ we therefore need $x_1 = x_2$.

$\Rightarrow$ All vectors of the form $\begin{pmatrix} t \\ t \end{pmatrix}$ for $t \in \mathbb{R}$ are orthogonal to all vectors in $\operatorname{im}(G)$.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}}_{[H]} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

(i) Calculate the image of H.

(ii) Decide if H is injective and/or surjective.

(iii) Find a linear map $J : \mathbb{R}^3 \to \mathbb{R}^4$ with $H(J(y)) = y$ for all $y \in \mathbb{R}^3$.

(iv) Show that there cannot exist a linear map $K : \mathbb{R}^3 \to \mathbb{R}^4$ with $K(H(x)) = x$ for all $x \in \mathbb{R}^4$.

(i) To calculate im(H) we need to find all $y \in \mathbb{R}^3$, s.th $H(x) = y$ has a solution.

$$([H] \mid y) = \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & y_1 \\ 0 & 1 & 1 & 0 & y_2 \\ 0 & 0 & 1 & 1 & y_3 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & y_1 \\ 0 & 1 & 0 & -1 & y_2 - y_3 \\ 0 & 0 & 1 & 1 & y_3 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & y_1 - y_2 + y_3 \\ 0 & 1 & 0 & -1 & y_2 - y_3 \\ 0 & 0 & 1 & 1 & y_3 \end{array}\right)$$

$\Rightarrow$ $H(x) = y$ has a solution for any $y \in \mathbb{R}^3$ (namely $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} y_1 - y_2 + y_3 \\ y_2 - y_3 \\ y_3 \\ 0 \end{pmatrix}$ is one)

$\Rightarrow$ $\text{im}(H) = \mathbb{R}^3$

(ii) By (i) we see that H <u>is surjective</u>. But since $H\begin{pmatrix}0\\0\\0\\0\end{pmatrix} = H\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ we see that H <u>is not injective</u>.

(iii) By (i) we get that $H(x) = y$ has a solution given by $x = \begin{pmatrix} y_1 - y_2 + y_3 \\ y_2 - y_3 \\ y_3 \\ 0 \end{pmatrix}$.

If we define $J : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \longmapsto \begin{pmatrix} y_1 - y_2 + y_3 \\ y_2 - y_3 \\ y_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

we therefore have $H(J(y)) = y$.

(iv) By (ii) we know that for $x = \begin{pmatrix}-1\\1\\-1\\0\end{pmatrix}$ we have $H(x) = 0$. For any linear map $K : \mathbb{R}^3 \to \mathbb{R}^4$ we would therefore get

$$K(H(x)) = K(0) = 0 \neq x = \begin{pmatrix}-1\\1\\-1\\0\end{pmatrix}.$$

Since this is always the case for a lin. map.