1) (10 Points) Consider the following linear system

$$\begin{cases} -x_1 + 2x_2 + 3x_3 + 2x_4 = 3 \\ 3x_1 - 2x_2 - x_3 + 2x_4 = 3 \\ x_1 + 2x_2 + 5x_3 + 6x_4 = 9 \end{cases}.$$

(i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

(ii) Calculate the row-reduced echelon forms of the matrices $(A \mid b)$ and A and calculate their ranks.

(iii) Determine all the solutions to the linear system $Ax = b$.

(iv) Find an injective linear map $F : \mathbb{R}^2 \to \mathbb{R}^4$ such that $Ax = 0$ for any $x \in \text{im}(F)$.

2) (8 Points) Let $u = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \in \mathbb{R}^2$ and define the following three functions:

$$f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1(u \bullet u) - x_2 \\ x_1 + x_3 \end{pmatrix},$$

$$f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto x_1 \sin(x_2),$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad x \longmapsto (x \bullet u)x.$$

(i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.

(ii) Is f_2 injective and/or surjective?

3) (8 Points) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \quad G\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

(i) Determine the matrix of G.

(ii) Find all vectors $x \in \mathbb{R}^2$ such that x is orthogonal to every vector $v \in \text{im}(G)$.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^4 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ x_3 + x_4 \end{pmatrix}.$$

(i) Calculate the image of H.

(ii) Decide if H is injective and/or surjective.

(iii) Find a linear map $J : \mathbb{R}^3 \to \mathbb{R}^4$ with $H(J(y)) = y$ for all $y \in \mathbb{R}^3$.

(iv) Show that there cannot exist a linear map $K : \mathbb{R}^3 \to \mathbb{R}^4$ with $K(H(x)) = x$ for all $x \in \mathbb{R}^4$.