# Linear Algebra I

Fall 2023

## Recall:

**Definition 8.1** A subset $U \subset \mathbb{R}^n$ is a <u>subspace</u> of $\mathbb{R}^n$ if

i) $0 \in \mathbb{R}^n$ (again: by $0$ we mean $\begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix}$ ↕ $n$ )

ii) For all $u, v \in U$ : $u+v \in U$ "U is closed under addition"

iii) Fo all $u \in U$ and $\lambda \in \mathbb{R}$: $\lambda u \in U$ "U is closed under scalar multiplication"

- The span of $v_1, \ldots, v_n \in \mathbb{R}^m$

$$\text{span}\{v_1, \ldots, v_n\} = \{\lambda_1 v_1 + \ldots + \lambda_n v_n \mid \lambda_1, \ldots, \lambda_n \in \mathbb{R}\}$$

is a subspace of $\mathbb{R}^m$.

- For lin. map. $F: \mathbb{R}^n \to \mathbb{R}^m$ the kernel & image are subspaces.

$$\{x \in \mathbb{R}^n \mid F(x) = 0\} = \ker(F) \subset \mathbb{R}^n \qquad \mathbb{R}^m \supset \text{im}(F) = \{y \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n, F(x) = y\}$$

If $[F] = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$ then $\text{im}(F) = \text{span}\{v_1, \ldots, v_n\}$

Fact: Every subspace is the kernel and image of some lin. maps.

Idea: If $U = \{x \in \mathbb{R}^n \mid \text{condition on } x\}$

try to find $F : \mathbb{R}^n \to \mathbb{R}^m$ with

condition on $x \iff F(x) = 0$.

Example: $U = \{x \in \mathbb{R}^3 \mid x_1 + x_2 = 0,\ x_1 - x_2 + x_3 = 0\}$

Set $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 + x_3 \end{pmatrix}$$

Then $\ker(F) = U$.

Theorem 8.7 Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.

i) The following statements are equivalent

$F$ is injective $\iff \ker(F) = \{0\} \iff \underset{\substack{\| \\ rk([F])}}{rk(F)} = n$.

ii) The following statements are equivalent

$F$ is surjective $\iff im(F) = \mathbb{R}^m \iff rk(F) = m$.

iii) If $m = n$ then the following statements are equivalent

$F$ is bijective $\iff$ $F$ is injective $\iff$ $F$ is surjective.

$\iff$ $F$ invertible

# § 9 Linear independence

We considered the linear map

$$G : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$$
$$x \longmapsto \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} x$$

and calculated its image

$$\operatorname{im}(G) = \mathbb{R}^2 = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

But we also learned that

$$\operatorname{im}(G) = \text{span of columns of } [G]$$
$$= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

This gives

$$\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

We just need 2 vectors

How can we show this?

Too much vectors. Can we remove some?

**Lemma 9.1** Let $v_1, \dots, v_\ell \in \mathbb{R}^m$. If $v_\ell \in \operatorname{span}\{v_1, \dots, v_{\ell-1}\}$ then

$$\underbrace{\operatorname{span}\{v_1, \dots, v_\ell\}}_{= V} = \underbrace{\operatorname{span}\{v_1, \dots, v_{\ell-1}\}}_{= W}.$$

**Proof:** Clearly we have $W \subset V$. Want to show $V \subset W$.

If $v \in \operatorname{span}\{v_1, \dots, v_\ell\} = V$, then there exist $\lambda_1, \dots, \lambda_\ell \in \mathbb{R}$ with

$$v = \lambda_1 v_1 + \dots + \lambda_\ell v_\ell. \qquad (*)$$

Since $v_\ell \in \operatorname{span}\{v_1, \dots, v_{\ell-1}\}$ there also exist $\alpha_1, \dots, \alpha_{\ell-1} \in \mathbb{R}$ with

$$v_\ell = \alpha_1 v_1 + \dots + \alpha_{\ell-1} v_{\ell-1}. \qquad (**)$$

Combining $(*)$ and $(**)$ gives

$$v = \lambda_1 v_1 + \dots + \lambda_{\ell-1} v_{\ell-1} + \lambda_\ell(\alpha_1 v_1 + \dots + \alpha_{\ell-1} v_{\ell-1})$$

$$= (\lambda_1 + \lambda_\ell \alpha_1) v_1 + \dots + (\lambda_{\ell-1} + \lambda_\ell \alpha_{\ell-1}) v_{\ell-1}$$

and therefore $v \in \underbrace{\operatorname{span}\{v_1, \dots, v_{\ell-1}\}}_{= W}$, i.e. $V \subset W$. $\square$

## Example 32

For the linear map $G$ in Example 16 we get

$$\operatorname{im}(G) = \operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\}$$

$\begin{pmatrix}2\\1\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \in \operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$

$\begin{pmatrix}1\\1\end{pmatrix} \in \operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$

$\parallel$ Lemma 9.1 since

$$\operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$$

$\parallel$ Lemma 9.1 $\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}$

$$\operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$$

$\parallel$ Lemma 9.1 $\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}$

$$\operatorname{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = \mathbb{R}^2.$$

**General question:** When is it possible to remove elements from $\operatorname{span}\{v_1, \dots, v_\ell\}$ without changing it?

**Definition 9.2** Vectors $v_1, \dots, v_\ell \in \mathbb{R}^m$ are called <u>linearly independent</u> if the equation

$$\lambda_1 v_1 + \dots + \lambda_\ell v_\ell = 0 \qquad (\lambda_1, \dots, \lambda_\ell \in \mathbb{R})$$

just has the <u>unique</u> solution $\lambda_1 = \lambda_2 = \dots = \lambda_\ell = 0$.

Otherwise $v_1, \dots, v_\ell$ are called <u>linearly dependent</u>.

## Example 33

Are the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix}$$

linearly independent?

The equation $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$ is equivalent to

$$\begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 5 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\begin{matrix} -2 & -1 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 5 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right) \sim \begin{matrix} -1 & 1 & -2 \end{matrix} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 6 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Solutions: $\lambda_1 = -2t$, $\lambda_2 = 3t$, $\lambda_3 = t$ $\Rightarrow$ $v_1, v_2, v_3$ are linearly dependent!

For $t=1$ we get $\lambda_1 = -2,\ \lambda_2 = 3,\ \lambda_3 = 1$

$$-2v_1 + 3v_2 + v_3 = -2\begin{pmatrix}1\\1\\2\end{pmatrix} + 3\begin{pmatrix}1\\-1\\1\end{pmatrix} + \begin{pmatrix}-1\\5\\1\end{pmatrix} = 0$$

$$\Rightarrow \quad v_3 = 2v_1 - 3v_2 \Rightarrow v_3 \in \operatorname{span}\{v_1, v_2\}$$

Lemma 9.1

$$\Rightarrow \quad \operatorname{span}\{v_1, v_2, v_3\} = \operatorname{span}\{v_1, v_2\}.$$

But $v_1, v_2$ are linearly independent since

$$\lambda_1 v_1 + \lambda_2 v_2 = 0 \rightsquigarrow \left(\begin{array}{cc|c}1 & 1 & 0\\1 & -1 & 0\\2 & 1 & 0\end{array}\right)$$

$$\wr$$

$$\left(\begin{array}{cc|c}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{array}\right).$$

$$\Rightarrow \lambda_1 = \lambda_2 = 0.$$

**Theorem 9.3** Let $v_1, \dots, v_\ell \in \mathbb{R}^m$. The following statements are equivalent:

i) $v_1, \dots, v_\ell$ are linearly dependent.

ii) There exists a $j = 1, \dots, \ell$ such that $v_j$ is a linear combination of the other vectors.

iii) There exists a $j = 1, \dots, \ell$ with

$$\operatorname{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_\ell\} = \operatorname{Span}\{v_1, \dots, v_\ell\}.$$

Proof: ii) $\Rightarrow$ iii) is Lemma 9.1

iii) $\Rightarrow$ ii) $v_j \in \operatorname{span}\{v_1, \dots, v_\ell\} \overset{\text{iii)}}{=} \operatorname{span}\{v_1, \dots, \cancel{v_j}, \dots, v_\ell\}$

$\Rightarrow v_j \in \operatorname{span}\{v_1, \dots, \cancel{v_j}, \dots, v_\ell\}$.

ii) $\Rightarrow$ i) Suppose $v_j = \lambda_1 v_1 + \dots + \lambda_{j-1} v_{j-1} + \lambda_{j+1} v_{j+1} + \dots + \lambda_\ell v_\ell$.

Then $0 = \lambda_1 v_1 + \dots + \lambda_{j-1} v_{j-1} \underbrace{- v_j}_{\lambda_j v_j \text{ with } \lambda_j = -1.} + \lambda_{j+1} v_{j+1} + \dots + \lambda_\ell v_\ell$.

$\Rightarrow v_1, \dots, v_\ell$ are linearly dependent.

i) $\Rightarrow$ ii): Suppose $\lambda_1 v_1 + \dots + \lambda_\ell v_\ell = 0$ with $\lambda_j \neq 0$.

Then $v_j = \left(\frac{\lambda_1}{\lambda_j}\right) v_1 + \dots + \left(\frac{\lambda_{j-1}}{\lambda_{j-1}}\right) v_{j-1} + \left(\frac{\lambda_{j+1}}{\lambda_j}\right) v_{j+1} + \dots + \left(\frac{\lambda_\ell}{\lambda_j}\right) v_\ell$

$\Rightarrow \quad v_j \in \operatorname{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_\ell\}$

$\square$

**Lemma 9.4** $V \subset \mathbb{R}^n$ subspace,
$v_1, \dots, v_\ell \in V$ linearly independent,
If $V = \operatorname{span}\{w_1, \dots, w_m\}$ for some $w_1, \dots, w_m \in \mathbb{R}^n$.
then $\ell \le m$.
(i.e. a space spanned by $m$ vectors can not contain more than $m$ linear independent vectors)

Proof: Homework 6

**Lemma 9.5** If $v_1, \dots, v_\ell \in \mathbb{R}^n$ are linearly independent and $w \in \mathbb{R}^n$ with $w \notin \operatorname{span}\{v_1, \dots, v_\ell\}$ then $v_1, \dots, v_\ell, w$ are linearly independent.

$(\lambda_1, \dots, \lambda_\ell, \mu \in \mathbb{R})$

Proof: Assume that $\lambda_1 v_1 + \dots + \lambda_\ell v_\ell + \mu w = 0$.

If $\mu \neq 0$, then $w = \left(\frac{\lambda_1}{\mu}\right) v_1 + \dots + \left(\frac{\lambda_\ell}{\mu}\right) v_\ell \in \operatorname{span}\{v_1, \dots, v_\ell\}$.

Hence $\mu = 0 \Rightarrow \lambda_1 v_1 + \dots + \lambda_\ell v_\ell = 0$

$$\Rightarrow \quad \lambda_1 = \dots = \lambda_\ell = 0$$

$v_1, \dots, v_\ell$ are lin. indep.

$$\Rightarrow \quad v_1, \dots, v_\ell, w \text{ are lin. indep.}$$

$\square$