Linear Algebra I Fall 2023

§ 8 Subspaces, Kernel & Image

In the previous lectures we considered subsets of $\mathbb{R}^n$, which arised when studying linear maps . For example, for a linear map $F: \mathbb{R}^n \to \mathbb{R}^m$ we calculated the image $\mathrm{im}(F) \subset \mathbb{R}^m$. In the case $m=3$ we saw that the image could be everything ($\mathbb{R}^3$), a plane, a line or just a point $0 \in \mathbb{R}^m$.

e.g. when $F: \mathbb{R}^3 \to \mathbb{R}^3$.
$x \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

These sets are examples of subspaces, which we define in a bit more abstract way by the following:

Definition 8.1 A subset $U \subset \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if

i) $0 \in \mathbb{R}^n$ (again: by 0 we mean $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Big\} n$)

ii) For all $u, v \in U$: $u + v \in U$ "U is closed under addition"

iii) Fo all $u \in U$ and $\lambda \in \mathbb{R}$: $\lambda u \in U$ "U is closed under scalar multiplication"

Example 29

1) $U = \{0\}$ and $U = \mathbb{R}^n$ are always subspaces of $\mathbb{R}^n$ for all $n \geq 1$.

2) Subspaces of $\mathbb{R}^n$:

$n=1$: $\{0\}$, $\mathbb{R}$

$n=2$: $\{0\}$, lines $\{\lambda v \mid \lambda \in \mathbb{R}\}$ $v \neq 0$, $v \in \mathbb{R}^2$, $\mathbb{R}^2$

$n=3$: $\{0\}$, lines, planes, $\mathbb{R}^3$

3) $U = \{x \in \mathbb{R}^3 \mid x_1 + x_2 - 3x_3 = 4\} \subset \mathbb{R}^3$ is <u>not</u> a subspace because $0 \notin U$.

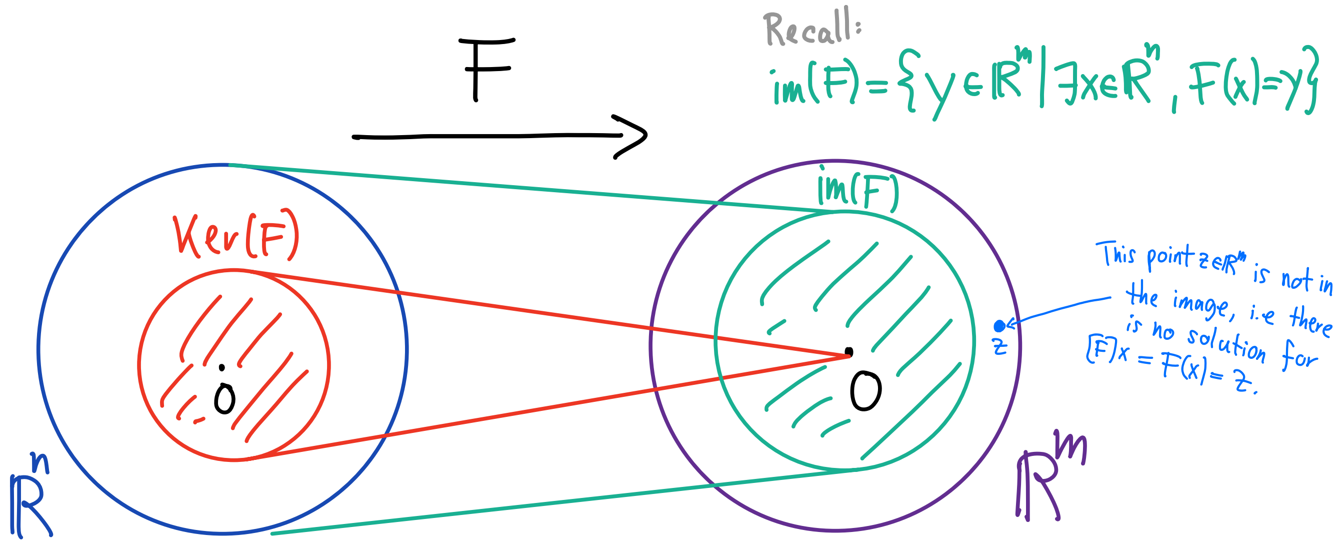
4) $U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid -1 \leq x_1 \leq 1 \right\}$ is also <u>not</u> a subspace. We have $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in U$ but $2 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \notin U$.

A lot of subspaces come from linear maps (actually all of them). We will see that the image of a linear map is a subspace. Another important space coming from a linear map is its kernel:

Definition 8.2 For a lin. map $F:\mathbb{R}^n \to \mathbb{R}^m$ the <u>Kernel of F</u> is defined by

$$\mathrm{Ker}(F) = \{ x \in \mathbb{R}^n \mid F(x) = 0 \}.$$

In other words: The Kernel of a linear map is the set of all solutions of the linear system $[F]x = 0$.



Proposition 8.3 Let $F:\mathbb{R}^n \to \mathbb{R}^m$ be a linear map.

i) The Kernel $\mathrm{Ker}(F)$ is a subspace of $\mathbb{R}^n$.

ii) The image $\mathrm{im}(F)$ is a subspace of $\mathbb{R}^m$

Proof: To show that a subset U is a subspace we need to check the 3 conditions

i) $0 \in \mathbb{R}^n$

ii) For all $u, v \in U$: $u+v \in U$

iii) Fo all $u \in U$ and $\lambda \in \mathbb{R}$: $\lambda u \in U$

Ker(F): i) We learned before that for any linear map we have $F(0)=0$.

$\Rightarrow \quad 0 \in \operatorname{Ker}(F)$.

ii) Let $u, v \in \operatorname{Ker}(F)$. Then we have

$$F(u+v) = F(u) + F(v) = 0 + 0 = 0$$

$\Rightarrow \quad u+v \in \operatorname{Ker}(F)$.

iii) Let $u \in \operatorname{Ker}(F)$ and $\lambda \in \mathbb{R}$ then

$$F(\lambda u) = \lambda F(u) = \lambda 0 = 0$$

$\Rightarrow \quad \lambda u \in \operatorname{Ker}(F)$.

$\Rightarrow \quad \operatorname{Ker}(F)$ is a subspace of $\mathbb{R}^n$.

im(F): i) Since $F(0)=0$ we have $0 \in \operatorname{im}(F)$.

ii) Let $u, v \in \operatorname{im}(F)$, i.e. $u = F(x)$, $v = F(y)$ for some $x, y \in \mathbb{R}^n$. Then we have

$$u+v = F(x) + F(y) = F(x+y)$$

$\Rightarrow \quad u+v \in \operatorname{im}(F)$.

ii) Let $u \in \mathrm{im}(F)$, $u = F(x)$, $\lambda \in \mathbb{R}$:

$$\lambda u = \lambda F(x) = F(\lambda x)$$

$$\Rightarrow \lambda u \in \mathrm{im}(F).$$

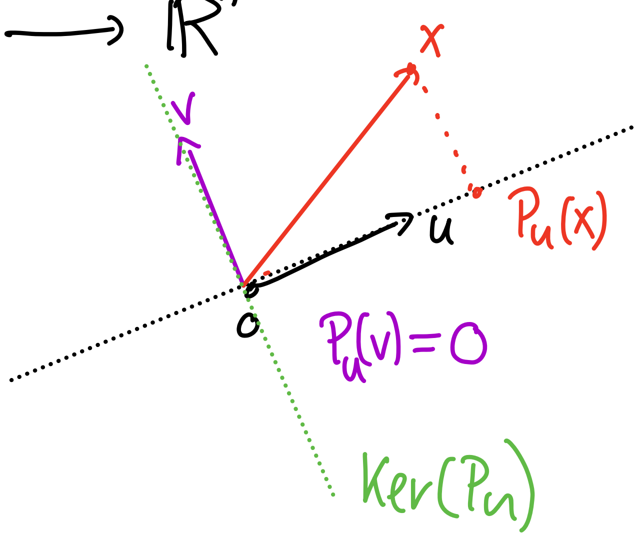
$\Rightarrow$ $\mathrm{im}(F)$ is a subspace of $\mathbb{R}^m$. $\square$

Fun fact: Actually every subspace can be written as the kernel and the image of some linear map. But this we can not prove yet.

Example 30

1) Let $u \in \mathbb{R}^n$, $u \neq 0$.

orthogonal proj.: $P_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$



The kernel of P_u is given by all vectors $v \in \mathbb{R}^n$ with $u \cdot v = 0$, because

$$P_u(v) = \frac{u \cdot v}{u \cdot u} u = 0.$$

$n = 2$: $\mathrm{Ker}(P_u) = $ line, $n = 3$: $\mathrm{Ker}(P_u) = $ plane

2) Consider the linear map

$$F: \mathbb{R}^2 \to \mathbb{R}^3$$
$$x \mapsto \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 1 \end{pmatrix} x$$

<u>Kernel:</u>

$x \in Ker(F) \Leftrightarrow F(x) = 0$

$$\left([F] \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right) \stackrel{(-2)}{=} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right) \stackrel{(1)}{\sim} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right) \stackrel{(-1)\,(1)}{\sim} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

$$\Rightarrow \quad x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \Rightarrow \quad Ker(F) = \{0\}.$$

<u>Image:</u> To calculate the image of F we need to check for which $y \in \mathbb{R}$ we have a $x \in \mathbb{R}^2$ with $F(x) = y$.

$$\left([F] \,\middle|\, \begin{matrix} y_1 \\ y_2 \\ y_3 \end{matrix}\right) \stackrel{(-2)}{=} \left(\begin{array}{cc|c} 1 & 1 & y_1 \\ 2 & 1 & y_2 \\ 0 & 1 & y_3 \end{array}\right) \stackrel{(1)}{\sim} \left(\begin{array}{cc|c} 1 & 1 & y_1 \\ 0 & -1 & y_2 - 2y_1 \\ 0 & 1 & y_3 \end{array}\right) \stackrel{(1)}{\sim} \left(\begin{array}{cc|c} 1 & 1 & y_1 \\ 0 & -1 & y_2 - 2y_1 \\ 0 & 0 & y_3 + y_2 - 2y_1 \end{array}\right)$$

$$\stackrel{(-1)}{\sim} \left(\begin{array}{cc|c} 1 & 0 & -y_1 + y_2 \\ 0 & -1 & -2y_1 + y_2 \\ 0 & 0 & -2y_1 + y_2 + y_3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -y_1 + y_2 \\ 0 & 1 & 2y_1 - y_2 \\ 0 & 0 & -2y_1 + y_2 + y_3 \end{array}\right)$$

$$\Rightarrow \quad im(F) = \left\{ y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, y_3 = 2y_1 - y_2 \right\}.$$

(Notice: This calculation can also be used to calculate the Kernel by setting $Y_1 = Y_2 = Y_3 = 0$.)

We can also write the image as follows:

$$\operatorname{im}(F) = \left\{ \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \;\middle|\; \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

3) Consider the linear map

$$G: \mathbb{R}^4 \longrightarrow \mathbb{R}^2$$

$$x \longmapsto \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} x$$

<u>Kernel:</u> Solve $[G]x = 0$

$$\left([G] \,\middle|\, \begin{matrix} 0 \\ 0 \end{matrix} \right) = \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array} \right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{array} \right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array} \right)$$

Solution:

$$x_1 = -2t_1 - t_2$$
$$x_2 = t_1$$

free var.

$$x_3 = t_1$$
$$x_4 = t_2$$

Another way of writing:

$$x = t_1 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore:

$$\operatorname{Ker}(G) = \left\{ t_1 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; t_1, t_2 \in \mathbb{R} \right\}.$$

Check yourself: $\operatorname{im}(G) = \mathbb{R}^2$

Definition 8.4 i) A <u>linear combination</u> of vectors $v_1, \dots, v_n \in \mathbb{R}^m$ is a vector of the form

$$u = \lambda_1 v_1 + \dots + \lambda_n v_n$$

for $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

ii) The <u>span</u> of $v_1, \dots, v_n \in \mathbb{R}^m$ is the set of all linear combinations:

$$\operatorname{span}\{v_1, \dots, v_n\} = \{\lambda_1 v_1 + \dots + \lambda_n v_n \in \mathbb{R}^m \mid \lambda_1, \dots, \lambda_n \in \mathbb{R}\}.$$

Example 31

1) $\begin{pmatrix}1\\2\\3\end{pmatrix}$ is a lin. comb. of $\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\3\end{pmatrix}$

since $\begin{pmatrix}1\\2\\3\end{pmatrix} = 2\begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}1\\0\\3\end{pmatrix}$.

$$\Longrightarrow \quad \begin{pmatrix}1\\2\\3\end{pmatrix} \in \operatorname{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\3\end{pmatrix}\right\}$$

2) Any $x = \begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$ is a lin. comb. of $\underset{\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}}{\underset{\parallel}{e_1}}, \dots, \underset{\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}}{\underset{\parallel}{e_n}}$:

$$x = x_1 e_1 + \dots + x_n e_n$$

and therefore $\mathbb{R}^n = \operatorname{span}\{e_1, \dots, e_n\}$.

3) In Example 16 we have

$$\operatorname{im}(F) = \operatorname{Span}\left\{\begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}$$

$$\operatorname{Ker}(G) = \operatorname{Span}\left\{\begin{pmatrix}-2\\1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\1\end{pmatrix}\right\}.$$

Notice: If $F: \mathbb{R}^n \to \mathbb{R}^m$, $x \mapsto Ax$ with $A = \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix}$

then $\operatorname{im}(F) = \operatorname{span}\{v_1, \dots, v_n\}$.

Proposition 8.5 For $v_1, \dots, v_n \in \mathbb{R}^m$ we have

i) $\operatorname{span}\{v_1, \dots, v_n\}$ is a subspace of $\mathbb{R}^m$.

ii) If $U \subset \mathbb{R}^m$ is a subspace and $v_1, \dots, v_n \in U$, then $\operatorname{span}\{v_1, \dots, v_n\} \subset U$.

Proof: Do yourself.

Recall: A linear map $F: \mathbb{R}^n \to \mathbb{R}^m$ is surjective if $\operatorname{im}(F) = \mathbb{R}^m$.

A similar statement exists for injective:

Proposition 8.6 A linear map $F: \mathbb{R}^n \to \mathbb{R}^m$ is injective if and only if $\operatorname{Ker}(F) = \{0\}$.

Proof: This is what you showed in Homework 3, Ex.3. $\square$

We also saw that F is injective if each column of $\operatorname{rref}([F])$ contains a pivot element, i.e. $\operatorname{rk}(F) = n$.
Similarly we saw F is surjective if each row contains a pivot element, i.e. $\operatorname{rk}(F) = m$.
Summarizing everything we get the following:

Theorem 8.7 Let $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map.

i) The following statements are equivalent

$$F \text{ is injective} \iff \ker(F) = \{0\} \iff \underset{\substack{\parallel \\ rk([F])}}{rk(F)} = n.$$

ii) The following statements are equivalent

$$F \text{ is surjective} \iff im(F) = \mathbb{R}^m \iff rk(F) = m.$$

iii) If $m=n$ then the following statements are equivalent

$$F \text{ is bijective} \iff F \text{ is injective} \iff F \text{ is surjective.}$$