Linear Algebra I

Fall 2023

Recall: $A \in \mathbb{R}^{\ell \times m}$, $B = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \in \mathbb{R}^{m \times n}$

$$A \cdot B = \begin{pmatrix} | & & | \\ Av_1 & \cdots & Av_n \\ | & & | \end{pmatrix} \in \mathbb{R}^{\ell \times n}$$

$I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathbb{R}^{n \times n}$ is the <u>Identity matrix</u>.

For all $v \in \mathbb{R}^n$ we have $I_n v = v$.

§7 The inverse of a linear map

Recall: (Lecture 3) $f: X \to Y$ is <u>invertible</u> if there exists a $\underline{g}: Y \to X$ with $g(f(x)) = x \quad \forall x \in X$

$f(g(y)) = y \quad \forall y \in Y$.

g: The inverse of f

Notation: $g = f^{-1}$

(equivalently: $g \circ f = id_X$, $f \circ g = id_Y$)

We saw: f is invertible $\Longleftrightarrow$ f is bijective

$$\underbrace{\forall y \in Y,}_{\text{surjective}} \underbrace{\exists!}_{\text{injective}} x \in X : y = f(x)$$

Define the Inverse by $g(y) = x$.

Questions:
- When is a linear map invertible?
- Is the inverse again linear?
- How to calculate the inverse?

Example 27: Consider the linear map

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Is F invertible ?

Take $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$ and check if $F(x) = y$ has a unique solution.

$$-2 \left(\begin{array}{cc|c} 1 & 3 & y_1 \\ 2 & 4 & y_2 \end{array}\right) \sim -\tfrac{1}{2} \left(\begin{array}{cc|c} 1 & 3 & y_1 \\ 0 & -2 & y_2 - 2y_1 \end{array}\right)$$

$$\sim -3 \left(\begin{array}{cc|c} 1 & 3 & y_1 \\ 0 & 1 & y_1 - \frac{1}{2} y_2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -2y_1 + \frac{3}{2} y_2 \\ 0 & 1 & y_1 - \frac{1}{2} y_2 \end{array}\right)$$

We get the unique solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2y_1 + \frac{3}{2} y_2 \\ y_1 - \frac{1}{2} y_2 \end{pmatrix} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

$\Rightarrow F$ is invertible.

The inverse of F:

$$F^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longmapsto \underbrace{\begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}}_{[F^{-1}].} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$\Rightarrow F^{-1}$ is linear.

Theorem 7.1 A linear map $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is invertible if and and only if $n = m = rk(F)$

$$\Longrightarrow \qquad \Longleftarrow \qquad \overset{\shortparallel}{} rk([F]).$$

Proof: F invertible $\iff$ $[F]x = y$ has a unique solution for all $y \in \mathbb{R}^m$

$$\begin{array}{c} \\ m \end{array} \overbrace{([F] \mid y)}^{n+1} \sim \dots \sim \underset{\substack{\shortparallel \\ rref([F]|y)}}{(B \mid z)}$$

$B = rref([F])$

$z \in \mathbb{R}^m$ can be an arbitrary vector depending on y.

"$\Longrightarrow$": Suppose F is invertible. Want to show $n = m = rk(F) = rk([F])$.

If $rk(F) < m$: $(B \mid z) = \begin{array}{c} \\ m \\ \\ \end{array} \left(\begin{array}{c|c} * & \\ \hline 0 & z \end{array} \right)$

$\Rightarrow$ No solutions for some z (i.e y).

$\Rightarrow$ $rk(F) = m$.

If $rk(F) < n$: $(B \mid z) = \overbrace{\left(\begin{array}{ccccc|c} 1 & & x & & & \\ & \ddots & x & & & \\ & & 1\; x & & & z \\ & & 0 & 1 & & \\ & & \vdots & & \ddots & \\ & & 0 & & & \end{array} \right)}^{n}$

$\Rightarrow$ There are columns without pivot element

$\Rightarrow$ No unique solution $\Rightarrow$ $rk(F) = n$.

"$\Leftarrow$": If $m = n = rk(F) \Rightarrow B = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$\Rightarrow$ $[F]x = y$ has a unique solution for all y

$\Rightarrow$ F is invertible. $\square$

Because of Thm. 7.1 we can assume $m=n$

Proposition 7.2: If a linear map $F: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is invertible, then F^{-1} is linear too.

Proof: Let $u, v \in \mathbb{R}^n$. Set $x = F^{-1}(u)$, $y = F^{-1}(v)$ i.e. $F(x) = u$, $F(y) = v$.

Then

- $x + y = \underbrace{F^{-1}F}_{id}(x+y) \overset{F \text{ lin.}}{=} F^{-1}(F(x) + F(y)) = F^{-1}(u+v)$
  $x + y = F^{-1}(u) + F^{-1}(v)$

$\lambda \in \mathbb{R}$

- $\lambda F^{-1}(u) = \lambda x = F^{-1}F(\lambda x) = F^{-1}(\lambda F(x)) = F^{-1}(\lambda u)$.

$\square$

Since F^{-1} is linear, there exists a matrix $[F^{-1}] \in \mathbb{R}^{n \times n}$

Definition 7.3 If $A \in \mathbb{R}^{n\times n}$ is the matrix of an invertible linear map F (i.e $[F]=A$), then we define the <u>inverse of A</u> by

$$A^{-1} := [F^{-1}].$$

Theorem 7.4 The inverse of A exists if and only if $\operatorname{rref}(A) = I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

We say <u>A is invertible</u> in this case.

<u>Proof</u>: Follows from the proof of Thm. 7.1.

Proposition 7.5 If $A, B \in \mathbb{R}^{n\times n}$ are invertible we have

i) $A A^{-1} = A^{-1} A = I_n$

ii) $(BA)^{-1} = A^{-1} B^{-1}$

$F: x \mapsto Ax$

$G: x \mapsto Bx$

Proof: (Check yourself)

i) $[F \circ F^{-1}] = A \cdot A^{-1} = A^{-1} \cdot A$

$\quad \| \qquad\qquad\qquad\qquad \|$

$[\mathrm{id}_{\mathbb{R}^n}] = I_n = [F^{-1} \circ F]$

ii) $BA = [G \circ F]$ and $(G\circ F)^{-1} = F^{-1} \circ G^{-1}$, therefore $(BA)^{-1} = [(G\circ F)^{-1}] = (F^{-1} \circ G^{-1}) = A^{-1} B^{-1}$.

How to determine A^{-1}?

In Example 27 we determined the inverse of $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ by solving the linear system $Ax = y$.

In general: If we want to determine the inverse of $A \in \mathbb{R}^{n \times n}$, we can try to bring the augmented matrix

$$(A \mid I_n) = \left(A \;\middle|\; \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}\right) \in \mathbb{R}^{2n \times 2n}$$

to rref.

Now we extend A by the matrix I_n. And not just a vector.

If A is invertible we will get

$$(A \mid I_n) \sim \cdots\cdots \sim \underbrace{(I_n \mid A^{-1})}_{= \operatorname{rref}(A \mid I_n)}.$$

Example 28: Determine the inverse of $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.

$$(A \mid I_2) = \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right) \begin{matrix} \\ \cdot\left(-\frac{1}{2}\right) \end{matrix}$$

$$\sim \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & -2 & \frac{3}{2} \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right) = (I_2 \mid A^{-1})$$

$$A^{-1} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$$

Check: $A \cdot A^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A^{-1}A$$

Compare this with Example 27.

<u>**Remark:**</u> 1) Row operations can also be described by multiplying with "elementary matrices". For each row operation (R1), (R2), (R3) there exist a corresponding elementary matrix. (See Def. 7.7 in lecture notes)

Definition 7.7 For $\lambda \in \mathbb{R}$ with $\lambda \neq 0$ and $1 \leq i, j \leq n$ we define the **elementary matrices** $R_i^{\lambda,j}, R_i^{\lambda}, R_{i,j} \in \mathbb{R}^{n\times n}$ by

$$R_i^{\lambda,j} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & \lambda & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}, \quad R_i^{\lambda} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \lambda & \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}, \quad R_{i,j} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & & 1 \\ & & & \ddots & \\ & & 1 & & 0 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}.$$

Here the λ in $R_i^{\lambda,j}$ is in the i-th row and j-th column, in R_i^{λ} it is in the i-th row, and in $R_{i,j}$ the 0 are on the diagonal in the i-th row and j-th column.

Multiplying with an elementary matrix from the left corresponds to the elementary row operations (Definition 2.6)

(R1) Multiyplying with $R_i^{\lambda,j}$: Add λ-times row j to row i.

(R2) Multiyplying with R_i^{λ}: Multiply row j by λ. ($\lambda \neq 0$)

(R3) Multiyplying with $R_{i,j}$: Change row i and j.

e.g.

$$\underset{\displaystyle A}{\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}} \; (-2) \sim \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix} \; \left(-\tfrac{1}{2}\right) \sim \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \; (-3) \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{rref}(A)$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \cdot = R_2^{-2,1} \qquad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \cdot = R_2^{-\frac{1}{2}} \qquad E_3 = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \cdot = R_1^{-3,2}$$

We get

$$\overbrace{E_3 \cdot \underbrace{E_2 \cdot E_1}_{\begin{pmatrix} 1 & 0 \\ 1 & -\frac{1}{2} \end{pmatrix}}}^{A^{-1}} \cdot A = I_2$$

$$\underbrace{\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & -\frac{1}{2} \end{pmatrix}}$$

$$A^{-1} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$$

Theorem 7.8 Every invertible matrix is a product of elementary matrices.

2) There are explicit formulas for the inverse of a matrix. (Cramer's rule)

For example for $n=2$ we have for $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Notice: This formula just makes sense for $ad-bc \neq 0$. Compare this to HW2 Ex 3 ii).

The determinant of A.

Cramer's rule gives a formula for all n, but it is not practical for real life applications. ("numerically expensive")