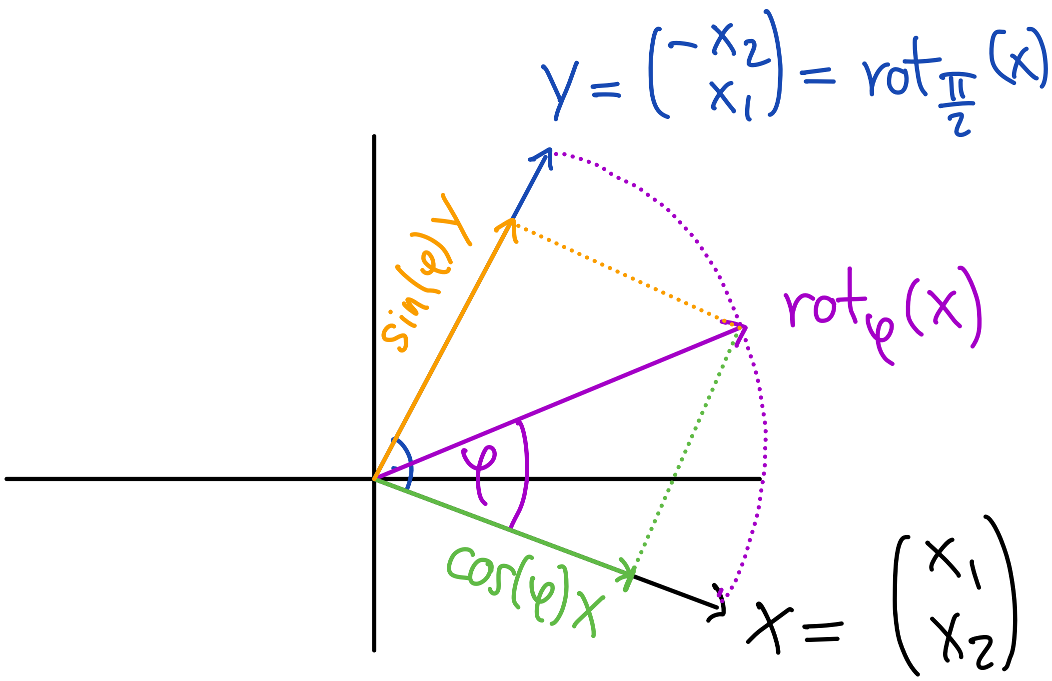
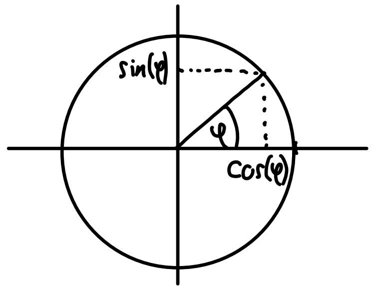
Linear Algebra I

Fall 2023

④ Rotations (in $\mathbb{R}^2$)

We want to describe a counterclockwise rotation with angle $\varphi \in \mathbb{R}$.





$$\begin{aligned}
\mathrm{rot}_\varphi(x) &= \cos(\varphi)\,x + \sin(\varphi)\,y \\
&= \cos(\varphi)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \sin(\varphi)\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\varphi)x_1 - \sin(\varphi)x_2 \\ \cos(\varphi)x_2 + \sin(\varphi)x_1 \end{pmatrix} \\
&= \underbrace{\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}}_{[\mathrm{rot}_\varphi]}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\end{aligned}$$

$$\Rightarrow \quad \mathrm{rot}_\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$x \longmapsto \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} x$$

is a linear map.

We have $\mathrm{rot}_{\varphi_1} \circ \mathrm{rot}_{\varphi_2} = \mathrm{rot}_{\varphi_1+\varphi_2}$, i.e.

rot_φ is invertible with inverse $\mathrm{rot}_{-\varphi}$:

$$\mathrm{rot}_\varphi \circ \mathrm{rot}_{-\varphi} = \mathrm{rot}_0 = \mathrm{id}.$$

(<u>Recall:</u> Tutorial 3: $E = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$.

For $\varphi = 0.927\ldots$ $(\approx 53^\circ)$

we have $[\mathrm{rot}_\varphi] = E$.)

§ 6 Composition of linear maps & Matrix multiplication

Linear maps are functions, so we can compose them

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "F"] \arrow[rr, bend right] & \mathbb{R}^m \arrow[r, "G"] & \mathbb{R}^\ell \end{tikzcd}$$

$$G \circ F = \underline{GF}$$

Notation we will use in the following.

<u>Theorem 6.1</u> If $F : \mathbb{R}^n \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^\ell$ are linear, then GF is linear.

Proof: For $x, y \in \mathbb{R}^n$ we have

$$GF(x+y) = G(F(x+y)) = G(F(x)+F(y))$$

Definition of GF

F is linear

$$= G(F(x)) + G(F(y))$$

G is linear

$$= GF(x) + GF(y).$$

For $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$:

$$GF(\lambda x) = G(F(\lambda x)) = G(\lambda F(x)) = \lambda G(F(x)) = \lambda GF(x).$$

$\square$

Question: What is the matrix of GF?

Example 22: We consider the following linear maps

$$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad\qquad G: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$x \longmapsto \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}}_{[F]} x \qquad\qquad x \longmapsto \underbrace{\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}}_{[G]} x$$

We want to calculate the matrix of $GF: \mathbb{R}^2 \to \mathbb{R}^3$.

$$[GF] = \begin{pmatrix} | & | \\ GF(e_1) & GF(e_2) \\ | & | \end{pmatrix}. \qquad \begin{pmatrix} e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$F(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad GF(e_1) = G(F(e_1)) = G\begin{pmatrix} 1 \\ 3 \end{pmatrix} = [G]\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$

$$F(e_2) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad GF(e_2) = G\left(F\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right) = [G]\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}.$$

$$\Longrightarrow [GF] = \begin{pmatrix} | & | \\ GF(e_1) & GF(e_2) \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ [G]\begin{pmatrix}1\\0\end{pmatrix} & [G]\begin{pmatrix}2\\-1\end{pmatrix} \\ | & | \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 3 & -1 \\ 1 & 2 \end{pmatrix}.$$

<u>**Definition 6.2**</u> Let $A \in \mathbb{R}^{l \times m}$, $B \in \mathbb{R}^{m \times n}$ with

$$B = \begin{array}{c c} & \\ m & \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \\ & n \end{array} \qquad (v_1, \dots, v_n \in \mathbb{R}^m).$$

Then we define the <u>product of A and B</u> by

$$A \cdot B = \begin{array}{c c} & \\ l & \begin{pmatrix} | & & | \\ Av_1 & \cdots & Av_n \\ | & & | \end{pmatrix} \\ & n \end{array} \in \mathbb{R}^{l \times n}.$$

$$A \cdot B \;/\!/\; AB$$

Need to be the same

$$\left(\begin{array}{c c} & m \\ l & \begin{pmatrix} | & \cdots & | \end{pmatrix} \\ & A \end{array} \cdot \begin{array}{c c} & n \\ m & \begin{pmatrix} | & \cdots & | \end{pmatrix} \\ & B \end{array} = \begin{array}{c c} & n \\ l & \begin{pmatrix} | & \cdots & | \end{pmatrix} \\ & AB \end{array} \right)$$

Example 23:

1) $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ then

$$A \cdot B = \begin{pmatrix} | & | \\ A\begin{pmatrix} 1 \\ 3 \end{pmatrix} & A\begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ | & | \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ 3 & -1 \\ 1 & 2 \end{pmatrix}.$$

Compare this with Example 10, where $[G] = A$, $[F] = B$ and $[GF] = A \cdot B$.

2) $$\begin{pmatrix} 0 & 3 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 1 & 1 \\ 3 & 2 \end{pmatrix}.$$

3) $$\underbrace{\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}}_{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Next Lecture: B is the inverse of A, $B = A^{-1}$.

Theorem 6.3 Let $F: \mathbb{R}^n \to \mathbb{R}^m$ and $G: \mathbb{R}^m \to \mathbb{R}^\ell$ be linear maps. Then

$$[G \circ F] = \underbrace{[GF]}_{\text{Matrix of the composition of the maps } F \text{ and } G.} = \underbrace{[G] \cdot [F]}_{\text{Product of the matrices of } G \text{ and } F.}$$

Proof: We have $[F] = \begin{pmatrix} | & & | \\ F(e_1) & \cdots & F(e_n) \\ | & & | \end{pmatrix}$

and

$$[G]\cdot[F] \overset{\text{Def. 6.2}}{=} \begin{pmatrix} | & & | \\ [G]F(e_1) & \cdots & [G]F(e_n) \\ | & & | \end{pmatrix}$$

$$\underset{G(x)=[G]x}{=} \begin{pmatrix} | & & | \\ G(F(e_1)) & \cdots & G(F(e_n)) \\ | & & | \end{pmatrix} = [GF].$$

$\square$

Example 24:

1) $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_1 - x_2 \\ x_1 + 3x_2 \end{pmatrix}$ $\qquad [F] = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$

What is the matrix of $F \circ F$?

By hand: $F(F(x)) = F\begin{pmatrix} 2x_1 - x_2 \\ x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} 2(2x_1 - x_2) - (x_1 + 3x_2) \\ (2x_1 - x_2) + 3(x_1 + 3x_2) \end{pmatrix}$

$$= \begin{pmatrix} 3x_1 - 5x_2 \\ 5x_1 + 8x_2 \end{pmatrix} \implies [FF] = \begin{pmatrix} 3 & -5 \\ 5 & 8 \end{pmatrix}.$$

Using Theorem 6.3: $[FF] = [F][F] = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ 5 & 8 \end{pmatrix}.$

2) At the beginning: $\mathrm{rot}_\varphi : \mathbb{R}^2 \to \mathbb{R}^2$

$$x \mapsto \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} x$$

is the rotation by angle φ.

$\mathrm{rot}_{\varphi_1} \circ \mathrm{rot}_{\varphi_2}$: rotation by φ_2 and then by φ_1

$$\Rightarrow \quad \mathrm{rot}_{\varphi_1} \circ \mathrm{rot}_{\varphi_2} \overset{(*)}{=} \mathrm{rot}_{\varphi_1+\varphi_2}.$$

$$[\mathrm{rot}_{\varphi_1+\varphi_2}] = \begin{pmatrix} \cos(\varphi_1+\varphi_2) & -\sin(\varphi_1+\varphi_2) \\ \sin(\varphi_1+\varphi_2) & \cos(\varphi_1+\varphi_2) \end{pmatrix}$$

Theorem 6.3:

$$[\mathrm{rot}_{\varphi_1} \circ \mathrm{rot}_{\varphi_2}] = [\mathrm{rot}_{\varphi_1}] \cdot [\mathrm{rot}_{\varphi_2}]$$

$$= \begin{pmatrix} \cos(\varphi_1) & -\sin(\varphi_1) \\ \sin(\varphi_1) & \cos(\varphi_1) \end{pmatrix} \begin{pmatrix} \cos(\varphi_2) & -\sin(\varphi_2) \\ \sin(\varphi_2) & \cos(\varphi_2) \end{pmatrix}$$

Def. 6.2

$$= \begin{pmatrix} \cos(\varphi_1)\cos(\varphi_2) - \sin(\varphi_1)\sin(\varphi_2) & * \\ \sin(\varphi_1)\cos(\varphi_2) + \cos(\varphi_1)\sin(\varphi_2) & * \end{pmatrix}$$

By (*) we obtain the angle sum identities:

$$\cos(\varphi_1+\varphi_2) = \cos(\varphi_1)\cos(\varphi_2) - \sin(\varphi_1)\sin(\varphi_2)$$

$$\sin(\varphi_1+\varphi_2) = \sin(\varphi_1)\cos(\varphi_2) + \cos(\varphi_1)\sin(\varphi_2).$$

Recall: $I_n = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} \in \mathbb{R}^{n \times n}$ is the

<u>Identity matrix</u>.

For all $v \in \mathbb{R}^n$ we have $I_n v = v$.

Proposition 6.4. For all $A \in \mathbb{R}^{l \times m}$, $B, D \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{n \times p}$, $\lambda \in \mathbb{R}$ we have

i) $A \cdot I_m = I_l \cdot A = A$.

ii) $(AB)C = A(BC)$

iii) $A(B+D) = AB + AD$

iv) $(B+D)C = BC + DC$

v) $\lambda(AB) = (\lambda A)B = A(\lambda B)$.

Proof: Check by yourself. Similar to HW2 Ex 1.

Remark: If $A, B \in \mathbb{R}^{n \times n}$ then in general we have $A \cdot B \neq B \cdot A$.

e.g. $\underset{A}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \cdot \underset{B}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ but $\underset{B}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} \cdot \underset{A}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

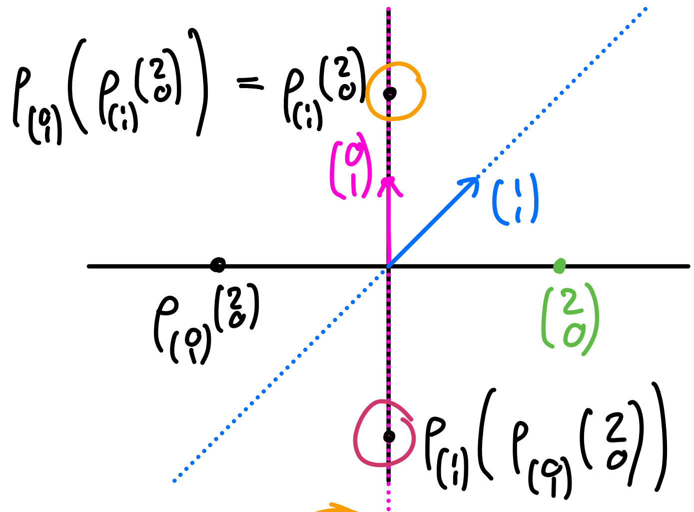
Example 25

(Not done in the lecture)

Recall: For $u \neq 0$ ρ_u denotes the reflection along the line spanned by u.

$$[\rho_{\binom{0}{1}}] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[\rho_{\binom{1}{1}}] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



$$\rho_{\binom{0}{1}}\left(\rho_{\binom{1}{1}}\binom{2}{0}\right) = \rho_{\binom{1}{0}}\binom{0}{2} = \binom{0}{2}$$

$$\neq$$

$$\rho_{\binom{1}{1}}\left(\rho_{\binom{0}{1}}\binom{2}{0}\right) = \rho_{\binom{1}{1}}\left(\binom{-2}{0}\right) = \binom{0}{-2}$$

$$[\rho_{\binom{0}{1}}]\cdot[\rho_{\binom{1}{1}}] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\neq$$

$$[\rho_{\binom{1}{1}}]\cdot[\rho_{\binom{0}{1}}] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\Longrightarrow$ Reflecting first along $\binom{1}{1}$ and then $\binom{0}{1}$ is different to first reflecting along $\binom{0}{1}$ and then $\binom{1}{1}$.

But also notice that sometimes (really rare) we have $A \cdot B = B \cdot A$.

e.g. $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.