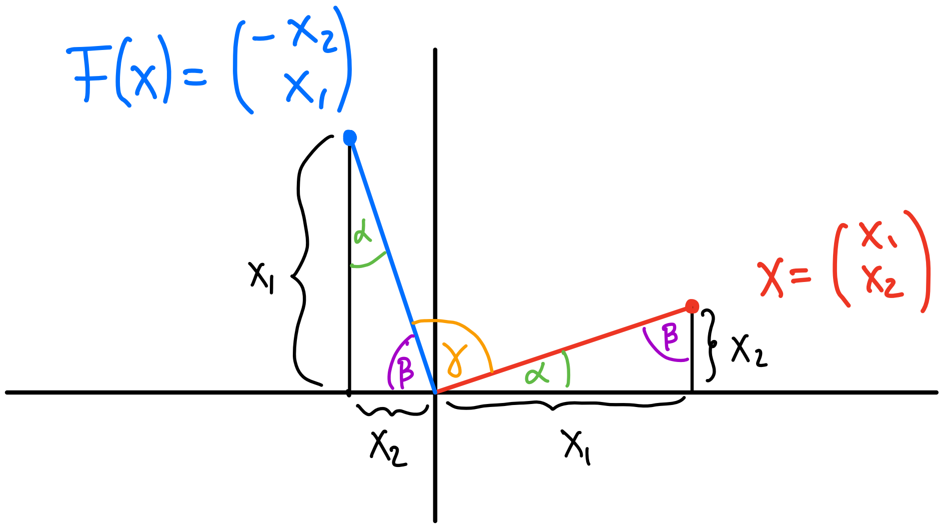
§ 5 Linear maps in geometry

Example 19

Consider the linear map $F: \mathbb{R}^2 \to \mathbb{R}^2$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}^{[F]} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



We have
$$\left.\begin{aligned} \alpha + \beta + \gamma &= 180^\circ \\ \alpha + \beta &= 90^\circ \end{aligned}\right\} \Rightarrow \gamma = 90^\circ.$$

$\Rightarrow$ F rotates x by 90° counterclockwise.

- $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $F(x) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$ are "orthogonal" to each other.

- How to check if $x, y \in \mathbb{R}^2$ are orthogonal in general? What about $x, y \in \mathbb{R}^n$?

Definition 5.1 Let $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$.

i) The <u>dot product</u> of u and v is defined by

$$u \bullet v = u_1 v_1 + \dots + u_n v_n \in \mathbb{R}$$

(vector $\bullet$ vector = number)

e.g. $\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \bullet \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = x_1(-x_2) + x_2 x_1 = 0 \right)$

ii) u and v are called <u>orthogonal</u> if $u \bullet v = 0$.

iii) The <u>norm</u> (= length) of u is defined by

$$\|u\| := \sqrt{u \bullet u} = \sqrt{u_1^2 + \dots + u_n^2}$$

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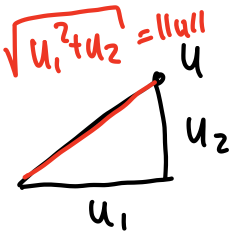
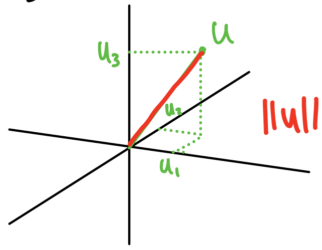
$n = 2$



$n = 3$



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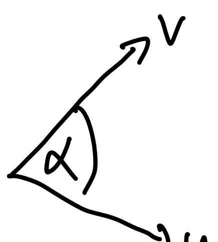
<u>Remark</u>: • The dot product (also called scalar product or inner product) allows us to speak about length and angles in $\mathbb{R}^n$. For $n = 2, 3$ one can show:



$$u \bullet v = \|u\| \cdot \|v\| \cdot \cos(\alpha).$$

For $n > 3$ this gives the definiton of an angle between u and v.

$T =$ "transpose"

- Write $u^T = (u_1 \; u_2 \; \cdots \; u_n) \in \mathbb{R}^{1\times n}$, then

$$\underset{\substack{\uparrow \\ \text{dot product}}}{u \bullet v} = \underset{\substack{\uparrow \\ \text{Matrix}\cdot\text{vector}}}{u^T v} = (u_1 \; u_2 \cdots u_n)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u_1v_1 + \dots + u_nv_n.$$

<u>Proposition 5.2</u> The dot product satisfies the following properties for $u, v, w \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$:

i) $u \bullet v = v \bullet u$

ii) $u \bullet (v + w) = u \bullet v + u \bullet w$

iii) $u \bullet (\lambda v) = \lambda (u \bullet v)$

Proof: Follows directly from the definition.

In the following we will give examples of linear maps which have a geometric interpretation.

① Scaling: Let $\lambda > 0$ and define the lin. map

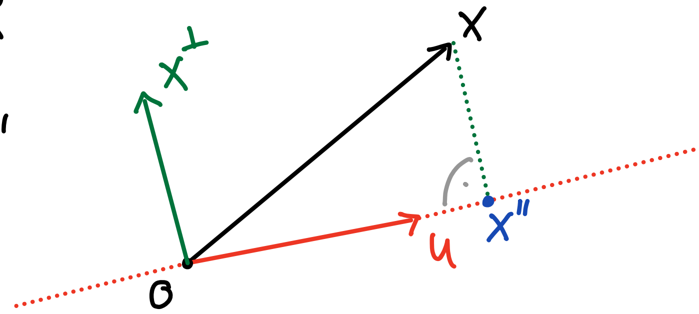
$$h_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$x \longmapsto \lambda x$$

$$[h_\lambda] = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix} = \lambda \underbrace{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{I_n} = \lambda I_n$$

Identity matrix.

② Orthogonal projection: Let $u \in \mathbb{R}^n$ with $u \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = 0$.

Want a map $P_u : \mathbb{R}^n \to \mathbb{R}^n$ that sends $x \in \mathbb{R}^n$ to $x^{\parallel}$, where $x = \underset{\text{orthogonal to } u}{x^{\perp}} + \underset{\text{parallel to } u}{x^{\parallel}}$



$$x^{\parallel} = \lambda u \quad \text{for some } \lambda \in \mathbb{R} \text{ we want to find.}$$

$$x^{\perp} \cdot u = 0$$

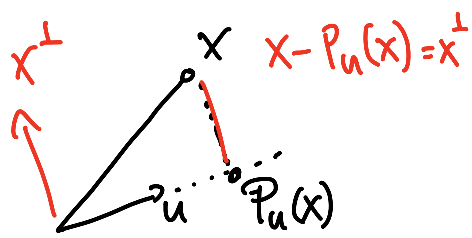
$$\rightsquigarrow \quad u \cdot x = u \cdot (x^{\perp} + x^{\parallel}) = \overbrace{u \cdot x^{\perp}}^{0} + u \cdot \overbrace{x^{\parallel}}^{\lambda u}$$

$$= u \cdot (\lambda u) = \lambda\, u \cdot u .$$

$$\Rightarrow \quad \lambda = \frac{u \cdot x}{u \cdot u} .$$

Definition 5.3 We define for $u \in \mathbb{R}^n$, $u \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = 0$

$$P_u : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto \frac{u \cdot x}{u \cdot u} u .$$



We get $\underbrace{(x - P_u(x))}_{= x^{\perp}} \cdot u = x \cdot u - \frac{u \cdot x}{u \cdot u} u \cdot u = 0.$

Proposition 5.4 P_u is a linear map.

Proof: For $x, y \in \mathbb{R}^n$ we have (using Prop 5.2)

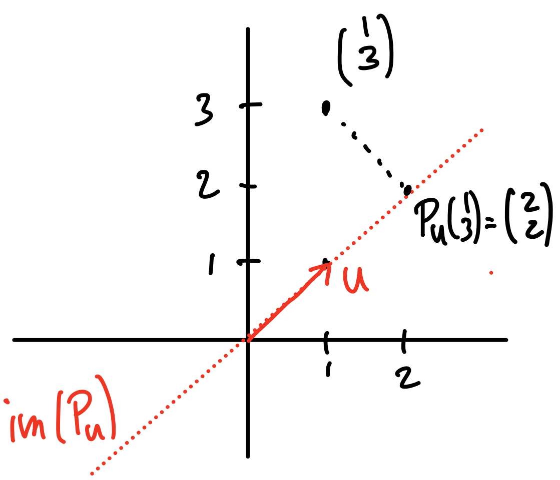
$$P_u(x+y) = \frac{u \cdot (x+y)}{u \cdot u} u = \frac{u \cdot x + u \cdot y}{u \cdot u} u$$

$$= \frac{u \cdot x}{u \cdot u} u + \frac{u \cdot y}{u \cdot u} u = P_u(x) + P_u(y).$$

$\lambda \in \mathbb{R}$.

$$P_u(\lambda x) = \frac{u \cdot (\lambda x)}{u \cdot u} u = \frac{\lambda\, u \cdot x}{u \cdot u} u = \lambda P_u(x).$$

Example 20: $n=2$, $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$



$$u \cdot u = 1 \cdot 1 + 1 \cdot 1 = 2$$

$$u \cdot x = 1 \cdot x_1 + 1 \cdot x_2 = x_1 + x_2$$

$$P_u(x) = \frac{u \cdot x}{u \cdot u} u = \frac{x_1 + x_2}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{x_1+x_2}{2} \\ \frac{x_1+x_2}{2} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}}_{} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$$[P_u] = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

e.g. $P_u \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

In the case $n=1$ the dot product is just the multiplication of real numbers and we have

$$P_u : \mathbb{R} \to \mathbb{R} \qquad (n=1)$$
$$x \mapsto \frac{u \cdot x}{u \cdot u} u = x,$$

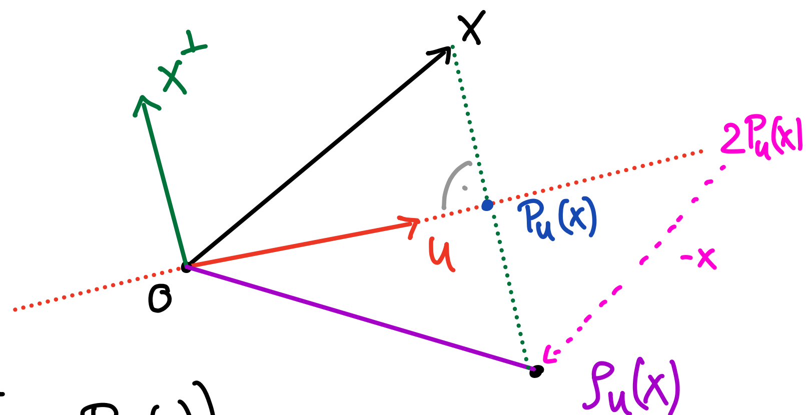
i.e $P_u = \mathrm{Id}$.

For $n > 1$ $P_u : \mathbb{R}^n \to \mathbb{R}^n$ is not injective and not surjective and

$$\mathrm{im}(P_u) = \{ \lambda \cdot u \mid \lambda \in \mathbb{R} \}.$$

③ Reflections

Now we want to reflect $x \in \mathbb{R}^n$ along the line spanned by u.



$$\rho_u(x) = x - 2x^\perp = x - 2(x - P_u(x))$$
$$= 2P_u(x) - x$$
$$x^\perp = x - P_u(x)$$
$$= 2\frac{u \cdot x}{u \cdot u} u - x$$

Definition 5.5 We define the reflection ρ_u along u by

$$\rho_u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$x \longmapsto 2\frac{u \cdot x}{u \cdot u} u - x.$$

Proposition 5.6 ρ_u is a linear map. (HW4)

Example 21: For $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ the ρ_u is the reflection along the diagonal.

What is the matrix of ρ_u?

$u \cdot e_1 = u \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$

$u \cdot u = 2$

We have $[\rho_u] = \begin{pmatrix} | & | \\ \rho_u(e_1) & \rho_u(e_2) \\ | & | \end{pmatrix}$

$$\rho_u(e_1) = \rho_u\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2\frac{u \cdot e_1}{u \cdot u} u - e_1 = u - e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

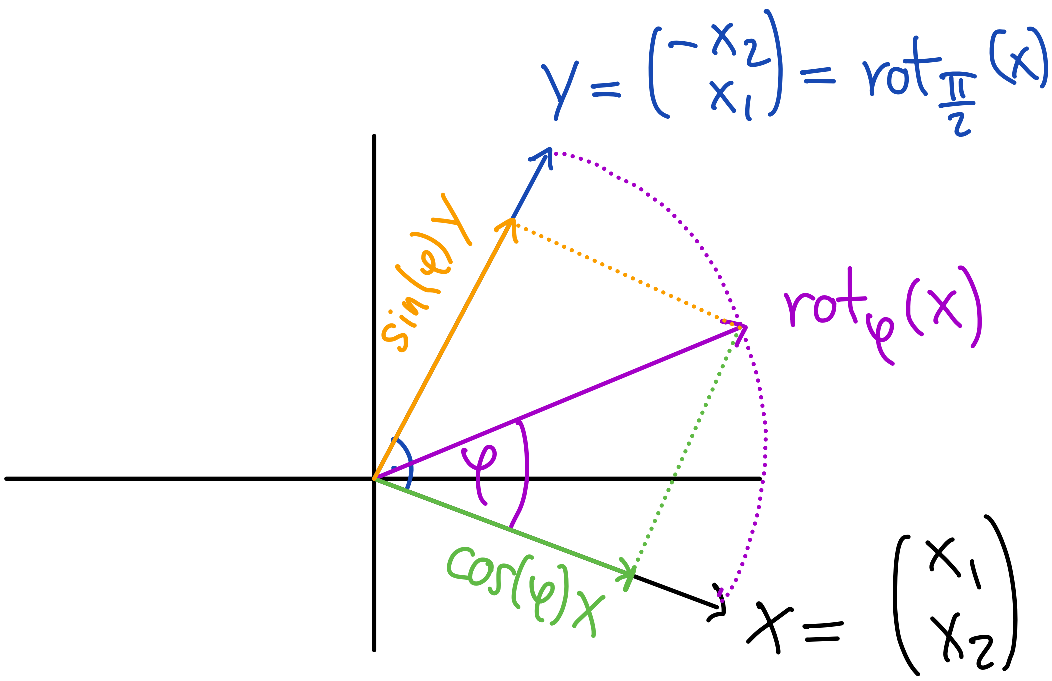
$$\rho_u(e_2) = 2\frac{u \cdot e_2}{u \cdot u} u - e_2 = u - e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

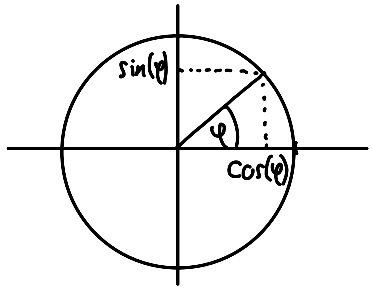
$$\Rightarrow \quad [\rho_u] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For any $n \geq 1$ and any $u \in \mathbb{R}^n$ the map ρ_u is bijective with ρ_u as its inverse. (HW4)

④ Rotations (in $\mathbb{R}^2$)

We want to describe a counterclockwise rotation with angle $\varphi \in \mathbb{R}$.





$$\mathrm{rot}_\varphi(x) = \cos(\varphi)x + \sin(\varphi)y$$

$$= \cos(\varphi)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \sin(\varphi)\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\varphi)x_1 - \sin(\varphi)x_2 \\ \cos(\varphi)x_2 + \sin(\varphi)x_1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}}_{[\mathrm{rot}_\varphi]}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\Rightarrow$ $\quad \mathrm{rot}_\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$x \longmapsto \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} x$$

is a linear map.

We have $\mathrm{rot}_{\varphi_1} \circ \mathrm{rot}_{\varphi_2} = \mathrm{rot}_{\varphi_1+\varphi_2}$, i.e. rot_φ is invertible with inverse $\mathrm{rot}_{-\varphi}$:

$$\mathrm{rot}_\varphi \circ \mathrm{rot}_{-\varphi} = \mathrm{rot}_0 = \mathrm{id}.$$

(Recall: Tutorial : $E = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$.

For $\varphi = 0.927\ldots$ $(\approx 53°)$

we have $[\mathrm{rot}_\varphi] = E$.)