Linear Algebra I

Fall 2023

Recall
Definition 3.3 A function $f: X \to Y$ is
i) injective if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.
in other words: For $y \in im(f)$ the equation $f(x)=y$ has a unique solution x.
ii) surjective if $im(f) = Y$ $(\forall y \in Y \, \exists x \in X: y = f(x))$
(image = codomain).
iii) bijective if it is injective and surjective.

Example 15 (continue)

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_1 + x_2 \\ x_2 - x_1 \end{pmatrix}$$

image of f?
surjective?
injective?

We want to calculate $im(f) = \{ y \in \mathbb{R}^3 \mid \exists x \in \mathbb{R}^2 : f(x) = y \}$.
For this we first rewrite

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x}$$

$\Rightarrow$ $im(f) = \{ y \in \mathbb{R}^3 \mid Ax = y$ has a solution $x \in \mathbb{R}^2 \}$.

Therefore we want to understand the solutions of $Ax = y$.

$$(A|y) = \left(\begin{array}{cc|c} 1 & 0 & y_1 \\ 1 & 1 & y_2 \\ -1 & 1 & y_3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & y_1 \\ 0 & 1 & y_2 - y_1 \\ 0 & 1 & y_3 + y_1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & y_1 \\ 0 & 1 & y_2 - y_1 \\ 0 & 0 & 2y_1 - y_2 + y_3 \end{array}\right)$$

This shows that $Ax = y$ has a solution iff $2y_1 - y_2 + y_3 = 0$
if and only if

$\Rightarrow$ $im(f) = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, 2y_1 - y_2 + y_3 = 0 \right\}$

There are no free variables

Since $im(f) \neq \mathbb{R}^3$ f is not surjective

For $y \in im(f)$ the system $Ax = y$ has a unique solution, i.e. f is injective.

§ 4 Linear maps

In this section we will be interested in a special family of functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ which play a major role in linear algebra.

Definition 4.1 A function $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a **linear map** if for all $u, v \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$

i) $F(u+v) = F(u) + F(v)$

ii) $F(\lambda u) = \lambda F(u)$

(We will define linear maps more generally in Linear Algebra 2, as maps $F: U \to V$, where U and V are vector spaces)

Example 16 1) For any $A \in \mathbb{R}^{m \times n}$ the function

$$F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$x \longmapsto Ax$$

is a linear map.

This follows from Proposition 2.4 (HW2, Ex 1):

i) $F(u+v) = A(u+v) = Au + Av = F(u) + F(v)$

ii) $F(\lambda u) = A(\lambda u) = \lambda A u = \lambda F(u)$.

In particular the function at the beginning is a linear map.

Special case: $n=m$: $F: \mathbb{R}^n \to \mathbb{R}^n$, $A = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = I_n$ identity matrix.

In this case $F(x) = x \quad \forall x \in \mathbb{R}$.

$\rightsquigarrow$ Identity map $id_{\mathbb{R}^n} = F$

2) The function $f: \mathbb{R}^2 \to \mathbb{R}^2$ is <u>not</u> linear (a linear map).

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \cdot x_2 \\ x_1 \end{pmatrix}$$

$\lambda = 2$, $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$f(\lambda \cdot u) = f\left(2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = f\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\lambda \cdot f(u) = 2 \cdot f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \cdot 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Therefore $f(\lambda u) \neq \lambda f(u)$.

In fact, we will see now that any linear map is given by a function like in Example 16 1).

Theorem 4.2 Let $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map, then there exists a unique matrix $[F] \in \mathbb{R}^{m\times n}$, such that

$$\underbrace{F(x)}_{\text{Evaluation of the function } F \text{ at } x} = \underbrace{[F]x}_{\text{Multiplication of the matrix } [F] \text{ with the vector } x.} \quad \text{for all } x \in \mathbb{R}^n.$$

Proof: For $1 \le j \le n$ we write $e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j \in \mathbb{R}^n$.

Every $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ can be (uniquely) written as

Scalar mult.

$$x = \underbrace{x_1 \cdot e_1}_{= \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}} + \underbrace{x_2 e_2}_{= \begin{pmatrix} 0 \\ x_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}} + \dots + \underbrace{x_n e_n}_{= \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_n \end{pmatrix}}.$$

$n=3$

$x = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$

$x = 3 \cdot e_1 - 1 \cdot e_2 + 4 e_3$
$= 3 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
$= \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$

Since F is linear, we have

$$F(x) = F(x_1 e_1 + x_2 e_2 + \dots + x_n e_n)$$

$$\overset{i)}{=} F(x_1 e_1) + F(x_2 e_2 + \dots + x_n e_n) = \dots = \underbrace{F(x_1 e_1)}_{x_1 F(e_1)} + \dots + F(x_n e_n)$$

$$\overset{ii)}{=} x_1 F(e_1) + \dots + x_n F(e_n).$$

Now set $[F] = \begin{pmatrix} | & | & & | \\ F(e_1) & F(e_2) & \dots & F(e_n) \\ | & | & & | \end{pmatrix} \in \mathbb{R}^{m \times n}.$

With this we have

$$[F]x = [F]\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 F(e_1) + x_2 F(e_2) + \dots + x_n F(e_n).$$

$$\left(\underset{A}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \underset{x}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} x_1 a + x_2 b \\ x_1 c + x_2 d \end{pmatrix} = x_1 \begin{pmatrix} a \\ c \end{pmatrix} + x_2 \begin{pmatrix} b \\ d \end{pmatrix} \right)$$

columns of A

And therefore $F(x) = [F]x$. $\square$

<u>Definition 4.3</u>: The matrix $[F]$ in Thm. 4.2 is called <u>the matrix of F</u>.

Notice: If F is a linear map and we know the values $F(e_j)$, $(1 \leq j \leq n)$ then we know the value of $F(x)$ for any x!

Example 17

1) If $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map with $F\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$ and $F\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$, then

$$F\begin{pmatrix}-1\\3\end{pmatrix} = F\left(-1\cdot\begin{pmatrix}1\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\end{pmatrix}\right) = -F\begin{pmatrix}1\\0\end{pmatrix} + 3F\begin{pmatrix}0\\1\end{pmatrix}$$

$$= -\begin{pmatrix}1\\2\end{pmatrix} + 3\begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}8\\10\end{pmatrix}$$

(In general: $F\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}1 & 3\\2 & 4\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$).

2) This works in more general. Assume that $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a linear map with

$$F\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix}, \quad F\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix}.$$

What is $F\begin{pmatrix}x_1\\x_2\end{pmatrix}$ for any x_1, x_2?

How to get this: Want a, b with $\underbrace{a\begin{pmatrix}1\\1\end{pmatrix} + b\begin{pmatrix}1\\0\end{pmatrix}}_{\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}} = \begin{pmatrix}x_1\\x_2\end{pmatrix}$

$$\rightsquigarrow \left(\begin{array}{cc|c}1 & 1 & x_1\\1 & 0 & x_2\end{array}\right) \sim \left(\begin{array}{cc|c}1 & 0 & x_2\\1 & 1 & x_1\end{array}\right) \sim \left(\begin{array}{cc|c}1 & 0 & x_2\\0 & 1 & x_1 - x_2\end{array}\right)$$

$\Rightarrow a = x_2,\ b = x_1 - x_2$

We have $\begin{pmatrix}x_1\\x_2\end{pmatrix} = x_2\begin{pmatrix}1\\1\end{pmatrix} + (x_1 - x_2)\begin{pmatrix}1\\0\end{pmatrix}$

$$\Rightarrow F\begin{pmatrix}x_1\\x_2\end{pmatrix} = F\left(x_2\begin{pmatrix}1\\1\end{pmatrix} + (x_1 - x_2)\begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$= x_2 F\begin{pmatrix}1\\1\end{pmatrix} + (x_1 - x_2) F\begin{pmatrix}1\\0\end{pmatrix}$$

$$= x_2\begin{pmatrix}0\\1\\2\end{pmatrix} + (x_1 - x_2)\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

$$= \begin{pmatrix} x_1 - x_2 \\ x_1 \\ -x_1 + 3x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ -1 & 3 \end{pmatrix}}_{[F]} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(We see: To know the value of a linear map F at x it suffices to know the value $F(v_1), \dots, F(v_n)$ where $v_1, \dots, v_n$ are vectors, such that we can write x as

$$x = \alpha_1 v_1 + \dots + \alpha_n v_n$$

for some $\alpha_1, \dots, \alpha_n \in \mathbb{R}$.)

How to check if a given function $F: \mathbb{R}^n \to \mathbb{R}^m$ is linear or not?

- To show that F is linear one can either show that there exist a matrix $A \in \mathbb{R}^{m \times n}$ with $F(x) = Ax$.
  OR one needs to show that **FOR ALL** $u, v \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ that $F(u+v) = F(u) + F(v)$ and $F(\lambda u) = \lambda F(u)$.

- To show that F is not linear, it suffices to give **ONE** example of $u, v \in \mathbb{R}^n$ with $F(u+v) \neq F(u) + F(v)$
  or **ONE** example of $u \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ with $F(\lambda u) \neq \lambda F(u)$.

For example the function $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto e^x$
is not linear, because the one explicit example $u = v = 0$ gives:

$$f(u+v) = f(0+0) = f(0) = e^0 = 1$$

$$f(u) + f(v) = f(0) + f(0) = e^0 + e^0 = 2$$

and therefore $f(u+v) \neq f(u) + f(v)$.

What you should not do:
Say "It is $e^{u+v} \neq e^u + e^v$ and therefore f is not linear."

Even though $e^{u+v} \neq e^u + e^v$ is true for almost all $u, v \in \mathbb{R}$, there are cases where it is not true.
e.g. for $u = 1$ and $v \approx 0.4586\ldots$

$$4.3 \approx e^{u+v} = e^u + e^v \approx 4.3.$$