Linear Algebra I

Fall 2023

Recall: Theorem 2.10 Every matrix A is row equivalent to a unique matrix B on row-reduced echelon form.
Notation: $B = \operatorname{rref}(A)$.

In Example 10: $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 4 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$(A \mid b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 1 & 4 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 5 & 3 \\ 0 & 1 & -1 & -1 \end{array}\right) = \operatorname{rref}(A|b)$$

Solutions of $Ax=b$ are $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$x_1 = 3 - 5t$$
$$x_2 = -1 + t$$
$$x_3 = t$$

Solving linear system $Ax=b \rightsquigarrow$ Find $\underline{\operatorname{rref}(A|b)} \overset{(B|C)}{=}$

General rref of an augmented matrix

$$\operatorname{rref}(A|b) = (B|c) = m \left(\begin{array}{cccccccccccccccc|c} 0 & \cdots & 0 & 1 & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * & 0 & * \\ & & & & & & & 1 & * & \cdots & * & 0 & * & \cdots & * & \vdots & \vdots \\ & & & & & & & & & & & 1 & * & \cdots & * & \vdots & \vdots \\ & & & & & & & & & & & & & \ddots & & 0 & \vdots \\ & & & & & & & & & & & & & & & 1 \; * \cdots & * \\ & & & & & & & & & & & & & & & & \lambda \end{array}\right) \in \mathbb{R}^{m \times (n+1)}$$

$$\underbrace{\qquad\qquad\qquad}_{n} \qquad \text{entries } * \text{ to } * \text{ of last column: } C' \qquad \lambda \in \{0,1\}$$

Reading off solutions for lin. system $Ax=b$: $\quad A, B \in \mathbb{R}^{m \times n}$, $b, c \in \mathbb{R}^m$

$$(A|b) \sim (B|c) = \operatorname{rref}(A|b)$$

1) If c contains a pivot element $(\lambda=1)$: No solutions

Else $(\lambda=0)$:

2) If every column of B contains a pivot element then we have the unique solution $x=c'$.

3) Some columns of B do not contain pivot element: Infinitely many solutions

Definition 2.11: The rank $rk(A)$ of a matrix $A\in\mathbb{R}^{m\times n}$ is the number of pivot elements in $rref(A)$.

Proposition 2.12: Let $A\in\mathbb{R}^{m\times n}$, $b\in\mathbb{R}^n$. The solutions of $Ax=b$ depend on $rk(A|b)$, $rk(A)$ as follows:

(i) $rk(A|b) > rk(A)$: No solutions

(ii) $rk(A|b) = rk(A)=n$: unique solution

(iii) $rk(A|b) = rk(A) < n$: Infinitely many solutions

Proof: Follows from the discussion before. (See lecture notes).

Examples: 1) $A=\begin{pmatrix}1&1\\0&0\end{pmatrix}$, $b=\begin{pmatrix}1\\1\end{pmatrix}$

2) $A=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $b=\begin{pmatrix}1\\1\end{pmatrix}$

3) $A=\begin{pmatrix}1&1\\1&1\end{pmatrix}$, $b=\begin{pmatrix}1\\1\end{pmatrix}$

§ 3 Sets & Functions

<u>Set</u> : A collection of distinct objects.

Precisely, but not necessariliy explicitlly, defined.

Already have seen examples of infinite sets: $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{m\times n}$.

Example 11 $\{2, 4, \pi\}$ finite set

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$ natural numbers

$\mathbb{Q}$: rational numbers

$\emptyset = \{\}$ empty set

If A is a set we write "$a \in A$" if a is an element of A and "$a \notin A$" if a is not an element of A.

$A \subset B$: A is a subset of B: If $a \in A$ then $a \in B$.

e.g. $$\emptyset \subset \underset{\underset{\{1,2,3\}}{\cup}}{\mathbb{N}} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

$\{a \in A \mid \text{condition}\}$: The (sub)set of all $a \in A$ that satisfy the condition.

e.g. $\{m \in \mathbb{N} \mid m \text{ is even}\}$

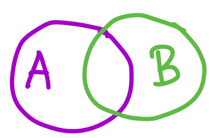
H : set of all humans

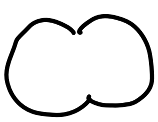
$NU = \{h \in H \mid h \text{ is a student at } NU\} \subset H$

$\{x \in \mathbb{R}^n \mid Ax = b\}$: solutions of $Ax=b$

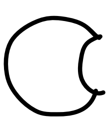
HW2: $\underset{A=\begin{pmatrix} a & b \\ c & a \end{pmatrix}}{L} = \{x \in \mathbb{R}^2 \mid Av = x \text{ for some } v \in \mathbb{R}^2\} \underbrace{= \mathrm{im}(A)}_{\text{later}}$

Operations on sets:



 Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

 Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

 Difference: $A \setminus B = \{x \in A \mid x \notin B\}$

e.g. $A = \{-1, 2, 3\}$, $B = \mathbb{N}$

$A \cup B = \{-1, 1, 2, 3, \dots\}$, $A \cap B = \{2, 3\}$,

$A \setminus B = \{-1\}$

Definition 3.1 X, Y: Sets

i) A function $f: X \to Y$ is a rule, assigning to each element $x \in X$ an element $f(x) \in Y$. This is also denoted by

$$\begin{aligned} f: X &\longrightarrow Y \\ x &\longmapsto f(x). \end{aligned}$$

ii) X: domain of f
Y: codomain of f

Functions are also called maps.

Definition 3.2 For a function $f: X \to Y$ the image of f is defined by

sometimes called "range" ran(f)

$$\mathrm{im}(f) = \{ y \in Y \mid \exists x \in X : y = f(x) \} \subset Y.$$

"there exists a" "with"

Definition 3.3

Composition of functions For two fct. $f: X \to Y$, $g: Y \to Z$ the composition $g \circ f$ of f and g is defined by

$$gf = g\circ f : X \longrightarrow Z$$
$$x \longmapsto (g\circ f)(x) = g(f(x)).$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Definition 3.4

A function $f: X \to Y$ is

i) <u>injective</u> if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

ii) <u>surjective</u> if $\mathrm{im}(f) = Y$ $\left(\forall y \in Y\ \exists x \in X : y = f(x)\right)$
(image = codomain).

iii) <u>bijective</u> if it is injective and surjective.

If f is bijective: For <u>every</u> $y \in Y$ there exists a
($f: X \to Y$) <u>unique</u> $x \in X$ with $f(x) = y$.

Then define $g: Y \longrightarrow X$ by $g(y) = x$ for $f(x) = y$.

Then $(g\circ f)(x) = g(f(x)) = g(y) = x$ $\forall x \in X \Rightarrow g\circ f = id_X$ identity on X

$(f\circ g)(y) = f(g(y)) = f(x) = y$ $\forall y \in Y \Rightarrow f\circ g = id_Y$ identity on Y.

That is, f is <u>invertible</u>, and g is <u>the inverse</u> of f.

Notation: $g = f^{-1}$.

Example 15

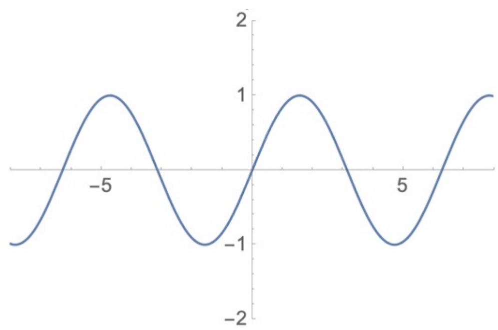
1) H : Set of all humans

$NU = \{ h \in H \mid h \text{ is a student at } NU \}$

$$f_1 : NU \longrightarrow \mathbb{N}$$
$$s \longmapsto \text{Student ID of student } s.$$

- f_1 is injective. (There are not two students with the same id.)
- f_1 is not surjective. (Not every number is the student ID of a student.)

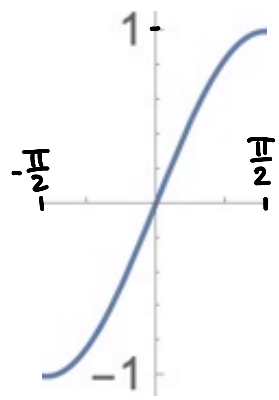
2)
$$f_2 : \underset{X}{\mathbb{R}} \longrightarrow \underset{Y}{\mathbb{R}}$$
$$x \longmapsto \sin(x)$$



- $\mathrm{im}(f_2) = [-1, 1] = \{ x \in \mathbb{R} \mid -1 \leq x \leq 1 \}$
- f_2 is not surjective. ($2 \in Y = \mathbb{R}$ but $2 \notin \mathrm{im}(f_2)$)
- f_2 is not injective. ($\underbrace{f_2(0)}_{=0} = \underbrace{f_2(2\pi)}_{=0}$ but $0 \neq 2\pi$).

3) $f_3 : \underbrace{[-\frac{\pi}{2}, \frac{\pi}{2}]}_{} \longrightarrow [-1, 1]$, with $X = [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $Y = [-1,1]$

$x \longmapsto \sin(x)$

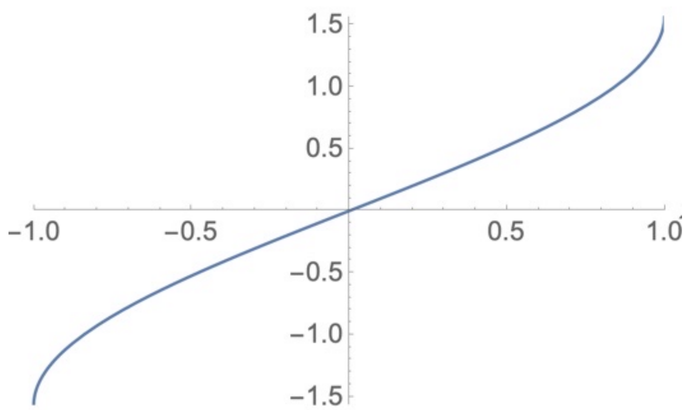


- $im(f_3) = [-1,1] = Y$

$\Rightarrow$ f_3 is surjective

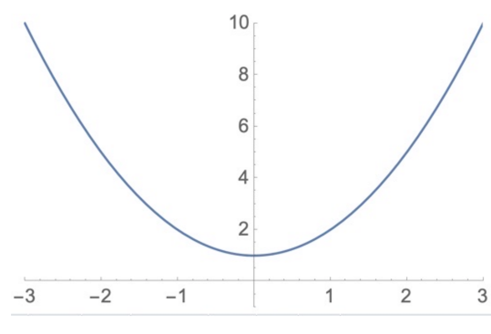
- f_3 is also injective: For each $y \in [-1,1] = Y$ there is <u>exactly one</u> $x \in [-\frac{\pi}{2}, \frac{\pi}{2}] = X$ with $f_3(x) = y$.

$\Rightarrow$ f_3 is bijective with inverse $f_3^{-1} = \arcsin$.



4) $f_4 : \mathbb{R} \longrightarrow \mathbb{R}$

$x \longmapsto x^2 + 1$



- $im(f_4) = \{x \in \mathbb{R} \mid x \geq 1\}$
- not injective and not surjective.

5) $$f_5 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{pmatrix} = \underbrace{\overset{A}{\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}} \overset{x}{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}}_{Ax}$$

To calculate the image of f_5 we need to find all $b = \binom{b_1}{b_2} \in \mathbb{R}^2$ such that there exists a $x = \binom{x_1}{x_2} \in \mathbb{R}^2$ with $Ax = b$.
$(f(x) = b)$

$$(A \mid b) = \left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 4 & b_2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 0 & b_2 - 2b_1 \end{array}\right)$$
(-2)

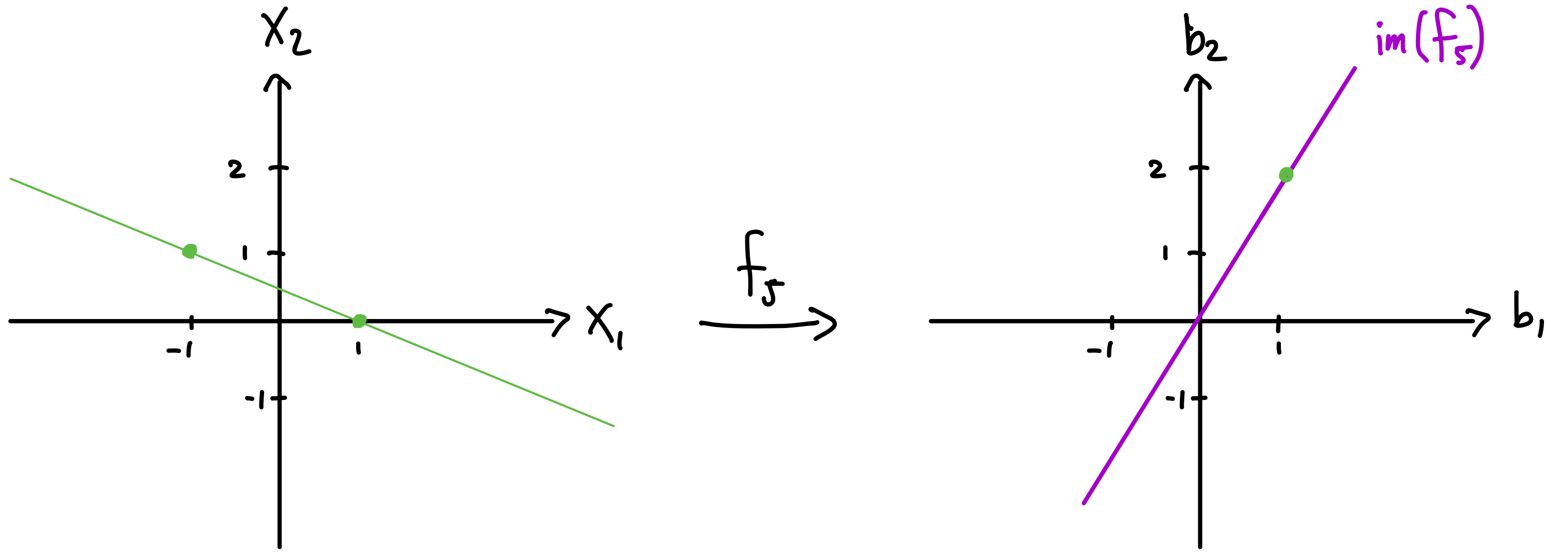
$\Rightarrow$ Just have a solution for $b_2 = 2b_1$.

$$\Longrightarrow \quad \mathrm{im}(f_5) = \left\{ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; b_2 = 2b_1 \right\}$$

If $b_2 = 2b_1$ we have the solution $x_1 = b_1 - 2t$, $x_2 = t$, $t \in \mathbb{R}$

$b_1 = 1$: $x_1 = 1 - 2t$, $x_2 = t$ and $f_5\begin{pmatrix} 1-2t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\forall t \in \mathbb{R}$

$\Longrightarrow$ f_5 is not injective since $\underset{t=0}{f_5\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \underset{t=1}{f\begin{pmatrix} -1 \\ 1 \end{pmatrix}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
but $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



6) $f_6 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{X}$$

Check for yourself:

- $\mathrm{im}(f_6) = \mathbb{R}^2$
- f_6 is bijective
- $f_6^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_1 + x_2 \\ \frac{3}{2}x_1 - \frac{1}{2}x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}}_{A^{-1}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

A^{-1}: Inverse of A.
Will discuss later in this course!