Linear Algebra I   Fall 2023

§2 Matrices & vectors

If you already saw matrices & vectors in school you should still carefully read the definitions.

Definition 2.1 i) A mxn-matrix is an array with m rows and n columns of numbers $a_{ij} \in \mathbb{R}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & \cdot & \cdots & a_{mn} \end{pmatrix} = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}} = (a_{ij})$$

We just write this if m, n are clear from context.

$\mathbb{R}^{m \times n}$: Set of all mxn-matrices

ii) A (column) vector of size n is a nx1-matrix

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}. \qquad v_1, \dots, v_n \in \mathbb{R}$$

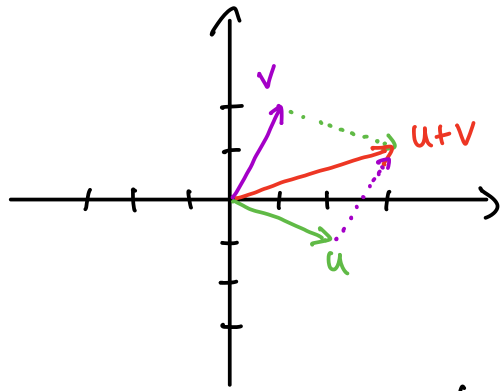
$\mathbb{R}^n$: Set of all vectors of size n.

$(= \mathbb{R}^{n \times 1})$

Example 8

For $n=2$ we can visualize vectors in the plane.

$$v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad u = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$



We can also add vectors, e.g. $u+v = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

In general the sum of matrices is defined by just adding each entry:

Definition 2.2 For $\overset{(a_{ij})}{\overset{\shortparallel}{A}}, \overset{(b_{ij})}{\overset{\shortparallel}{B}} \in \mathbb{R}^{m\times n}$ and $\lambda \in \mathbb{R}$ we define:

$$A + B = (a_{ij} + b_{ij}) \in \mathbb{R}^{m\times n} \qquad \text{(sum of matrices).}$$

$$\lambda A = (\lambda\, a_{ij}) \in \mathbb{R}^{m\times n} \qquad \text{(scalar multiplication).}$$

In the case $\lambda = -1$ we write $-A = (-1)A$.

Special case: vectors

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \qquad u+v = \begin{pmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{pmatrix}, \quad \lambda v = \begin{pmatrix} \lambda v_1 \\ \vdots \\ \lambda v_n \end{pmatrix}.$$

Warning: Whenever you see "+" you now should be aware if this is the addition of matrices/vectors or the usual addition of real numbers.

Definition 2.3 The product of a matrix $A=(a_{ij})\in\mathbb{R}^{m\times n}$ and a vector $v\in\mathbb{R}^n$ is defined by

$$Av=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & \cdot & \cdots & a_{mn}\end{pmatrix}\begin{pmatrix} v_1 \\ \vdots \\ v_n\end{pmatrix}=\begin{pmatrix} a_{11}v_1+a_{12}v_2+\dots+a_{1n}v_n \\ \vdots \\ a_{m1}v_1+a_{m2}v_2+\dots+a_{mn}v_n\end{pmatrix}.$$

Av is just defined if # columns of A = size of v.

Proposition 2.4 For $A\in\mathbb{R}^{m\times n}$, $x,y\in\mathbb{R}^n$ and $\lambda\in\mathbb{R}$ we have

i) $A(x+y)=Ax+Ay$.

ii) $A(\lambda x)=\lambda(Ax)$.

Proof: Homework 2.

Example 9 $\begin{pmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{pmatrix}\cdot\begin{pmatrix}-1\\ 3\end{pmatrix}=\begin{pmatrix}1\cdot(-1)+4\cdot 3\\ 2\cdot(-1)+5\cdot 3\\ 3\cdot(-1)+6\cdot 3\end{pmatrix}=\begin{pmatrix}11\\ 13\\ 15\end{pmatrix}.$

Example 10:

Let $A=\begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 4\end{pmatrix}$, $b=\begin{pmatrix}1\\ 2\end{pmatrix}$.

Find all $x=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}\in\mathbb{R}^3$ with $Ax=b$.

$$Ax=\begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}x_1+2x_2+3x_3\\ x_1+x_2+4x_3\end{pmatrix}=\begin{pmatrix}1\\ 2\end{pmatrix}=b$$

$$\Leftrightarrow \begin{cases} x_1+2x_2+3x_3=1\\ x_1+x_2+4x_3=2\end{cases}$$

This is a linear system. We also call $Ax=b$ a linear system.

Solving:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ x_1 + x_2 + 4x_3 = 2 \end{cases} \quad (-1) \iff \begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ \quad\;\; -x_2 + x_3 = 1 \end{cases} \quad (2)$$

$$\iff \begin{cases} x_1 \qquad\;\; + 5x_3 = 3 \\ \quad -x_2 + x_3 = 1 \end{cases} \quad (-1) \iff \begin{cases} x_1 \qquad\;\; + 5x_3 = 3 \\ \quad x_2 - x_3 = -1 \end{cases}$$

Pivot

free

Solutions:

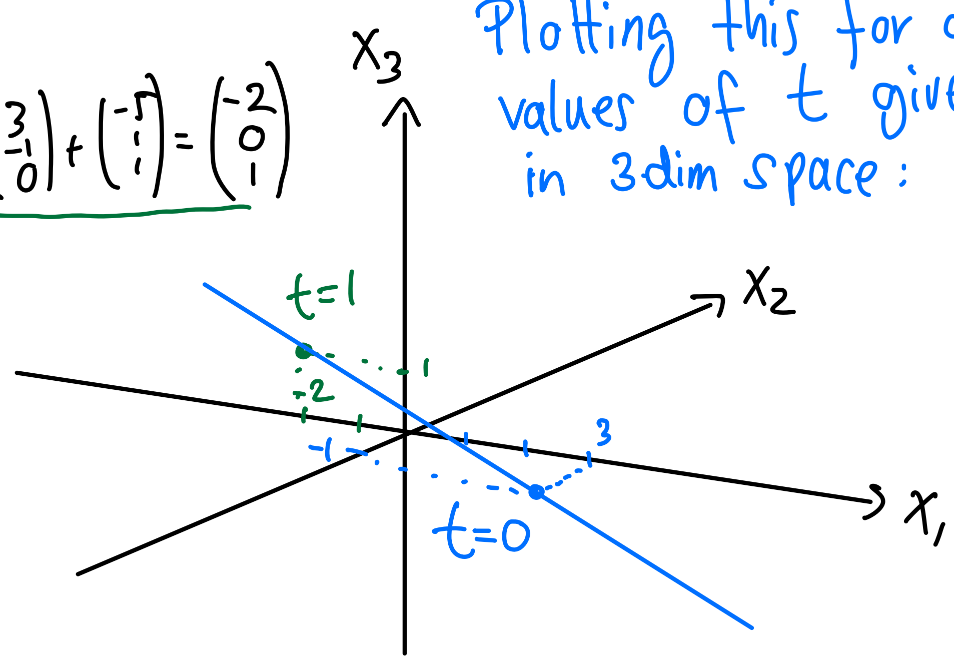
$$\begin{aligned} x_1 &= 3 - 5 \cdot t \\ x_2 &= -1 + t \qquad \text{for } t \in \mathbb{R} \\ x_3 &= t \end{aligned}$$

Using the vector notation this can be written as:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 - 5 \cdot t \\ -1 + t \\ t \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}.$$

Plotting this for all possible values of t gives a line in 3dim space:

$t = 1$: $x = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$



Matrix notation for linear systems:

Definition 2.5 For a $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ the matrix

$$(A \mid b) = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right) \in \mathbb{R}^{m \times (n+1)}$$

This line has no real meaning and is just for a better understanding

is called the <u>augmented matrix</u> of the linear system $Ax = b$.

In the previous example: $(A|b) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 1 & 1 & 4 & 2 \end{array}\right)$.

Solving a linear system $\rightsquigarrow$ apply row operations on the augmented matrix.

Definition 2.6 <u>Elementary row operations</u> on a matrix:

(R1) Add a multiple of one row to another one.

(R2) Multiply a row with a non-zero number.

(R3) Interchange two rows.

Definition 2.7 Two matrices A and B are called <u>row equivalent</u>, if B can be obtained from A by elementary row operations.

Notation: $A \sim B$.

In Example 10 (Matrix notation for solving lin. system)

$$(A|b) = \underset{R1}{\begin{array}{c}(-1)\\ \end{array}} \left(\begin{array}{ccc|c}1&2&3&1\\1&1&4&2\end{array}\right) \sim \underset{R1}{\begin{array}{c}\\ (2)\end{array}} \left(\begin{array}{ccc|c}1&2&3&1\\0&-1&1&1\end{array}\right)$$

$$\sim \underset{R2}{\begin{array}{c}\\ (-1)\end{array}} \left(\begin{array}{ccc|c}1&0&5&3\\0&-1&1&1\end{array}\right) \sim \left(\begin{array}{ccc|c}1&0&5&3\\0&1&-1&-1\end{array}\right).$$

Proposition 2.8 Let $A, B \in \mathbb{R}^{m \times n}$ and $b, c \in \mathbb{R}^m$. If $(A|b) \sim (B|c)$ then the linear systems $Ax=b$ and $Bx=c$ have the same solutions.

Definition 2.9 A matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ is on <u>row-reduced echelon form</u> (rref) if

i) The first non-zero element on each row (if any) is equal 1.

ii) If there is a leading 1 in a row, then all rows above contain a leading 1 further to the left.

iii) If a_{ij} is the first non-zero element in row i, then there are no other non-zero elements in the j-th column.

The first non-zero element in a row of a matrix on rref is called <u>pivot element</u>.

Theorem 2.10 Every matrix A is row equivalent to a unique matrix B on row-reduced echelon form.

Notation: $B = \mathrm{rref}(A)$.

"Proof": Induction on the number of columns. $(B|C)$ "

we learn this later.

Solving linear system $Ax=b \rightsquigarrow$ Find $\underline{\mathrm{rref}(A|b)}$

General rref of an augmented matrix

$$\mathrm{rref}(A|b) = (B|c) = m\left\updownarrow \left(\begin{array}{ccccccccccccccccc|c} 0 & \cdots & 0 & 1 & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots & * & 0 & & * \\ & & & & & & & 1 & * & \cdots & * & 0 & * & \cdots & * & \vdots & & \vdots \\ & & & & & & & & & & & 1 & * & \cdots & * & \vdots & & \vdots \\ & & & & & & & & & & & & & \ddots & & 0 & & \vdots \\ & & & & & & & & & & & & & & & 1 & * \cdots & * \\ & & & & & & & & & & & & & & & & & \lambda \end{array}\right)\right. \begin{array}{l} \left.\vphantom{\begin{array}{c}*\\ \vdots\\ \vdots\\ \vdots\\ *\end{array}}\right\} c' \\ \\ \end{array}$$

$$\longleftarrow n \longrightarrow$$

$(B|c) \in \mathbb{R}^{m\times(n+1)}$, $\lambda \in \{0,1\}$

Reading off solutions for lin. system $Ax=b$:

$$(A|b) \sim (B|c) = \mathrm{rref}(A|b) \qquad A, B \in \mathbb{R}^{m\times n},\quad b, c \in \mathbb{R}^m$$

1) If c contains a pivot element $(\lambda = 1)$: No solutions

Else $(\lambda = 0)$:

2) If every column of B contains a pivot element then we have the unique solution $x = c'$.
3) Some columns of B do not contain pivot element: Infinitely many solutions

Definition 2.11: The <u>rank</u> $rk(A)$ of a matrix $A \in \mathbb{R}^{m\times n}$ is the number of pivot elements in $rref(A)$.

Proposition 2.12: Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^n$. The solutions of $Ax=b$ depend on $rk(A|b)$, $rk(A)$ as follows:

(i) $rk(A|b) > rk(A)$: No solutions

(ii) $rk(A|b) = rk(A) = n$: unique solution

(iii) $rk(A|b) = rk(A) < n$: Infinitely many solutions

Proof: Follows from the discussion before. (See lecture notes).