Linear Algebra I

Fall 2023

§ 11 Coordinates

From now on we will considere ordered bases. This means we will write $(b_1, \dots, b_1)$ for a basis $\{b_1, \dots, b_n\}$. The difference is that $\{b_1, b_2\} = \{b_2, b_1\}$ but $(b_1, b_2) \neq (b_2, b_1)$.

Definition 11.1 Let $B = (b_1, \dots, b_m)$ be a basis of $V \subset \mathbb{R}^n$. We define the <u>coordinate map</u> by

$$C_B : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} \longmapsto \lambda_1 b_1 + \dots + \lambda_m b_m.$$

Theorem 11.2 Let $B = (b_1, \dots, b_m)$ be a basis of $V \subset \mathbb{R}^n$.

i) $C_B : \mathbb{R}^m \longrightarrow V$ is bijective.

ii) <u>For all $x \in V$ there exist unique $\lambda_1, \dots, \lambda_m$</u> $\in \mathbb{R}$ with

$(\forall x \in V \; \exists! \lambda_1, \dots, \lambda_m \in \mathbb{R})$

$$x = \lambda_1 b_1 + \dots + \lambda_m b_m.$$

Proof: i) Since $\operatorname{im} c_B = \operatorname{span}\{b_1, \dots, b_m\} = V$
the map $c_B : \mathbb{R}^m \to V$ is surjective.

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} = \lambda \in \ker c_B \iff \lambda_1 b_1 + \dots + \lambda_m b_m = 0$$

$$\underset{\substack{\uparrow \\ b_1, \dots, b_m \text{ lin. indep.}}}{\Longrightarrow} \lambda = 0 \Rightarrow \ker c_B = \{0\}$$

$$\underset{\text{Thm. 8.7.}}{\iff} c_B \text{ injective}$$

$\Rightarrow$ c_B bijective.

ii) This is just a reformulation of i).

Definition 11.3 Let $B = (b_1, \dots, b_m)$ be a basis of $V \subset \mathbb{R}^n$ and $x \in V$ with $x = \lambda_1 b_1 + \dots + \lambda_m b_m$.

i) The number $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ are the coordinates of x (in the basis B).

ii) The coordinate vector of x is

$$[x]_B = c_B^{-1}(x) = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}.$$

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1) $\mathcal{B} = (e_1, \dots, e_n)$ is a basis of $\mathbb{R}^n$.

For all $x \in \mathbb{R}^n$ we have $[x]_{\mathcal{B}} = x$.

2) Consider $b_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $b_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

$\mathcal{B} = (b_1, b_2)$ is a basis of $V = \operatorname{span}\{b_1, b_2\}$.

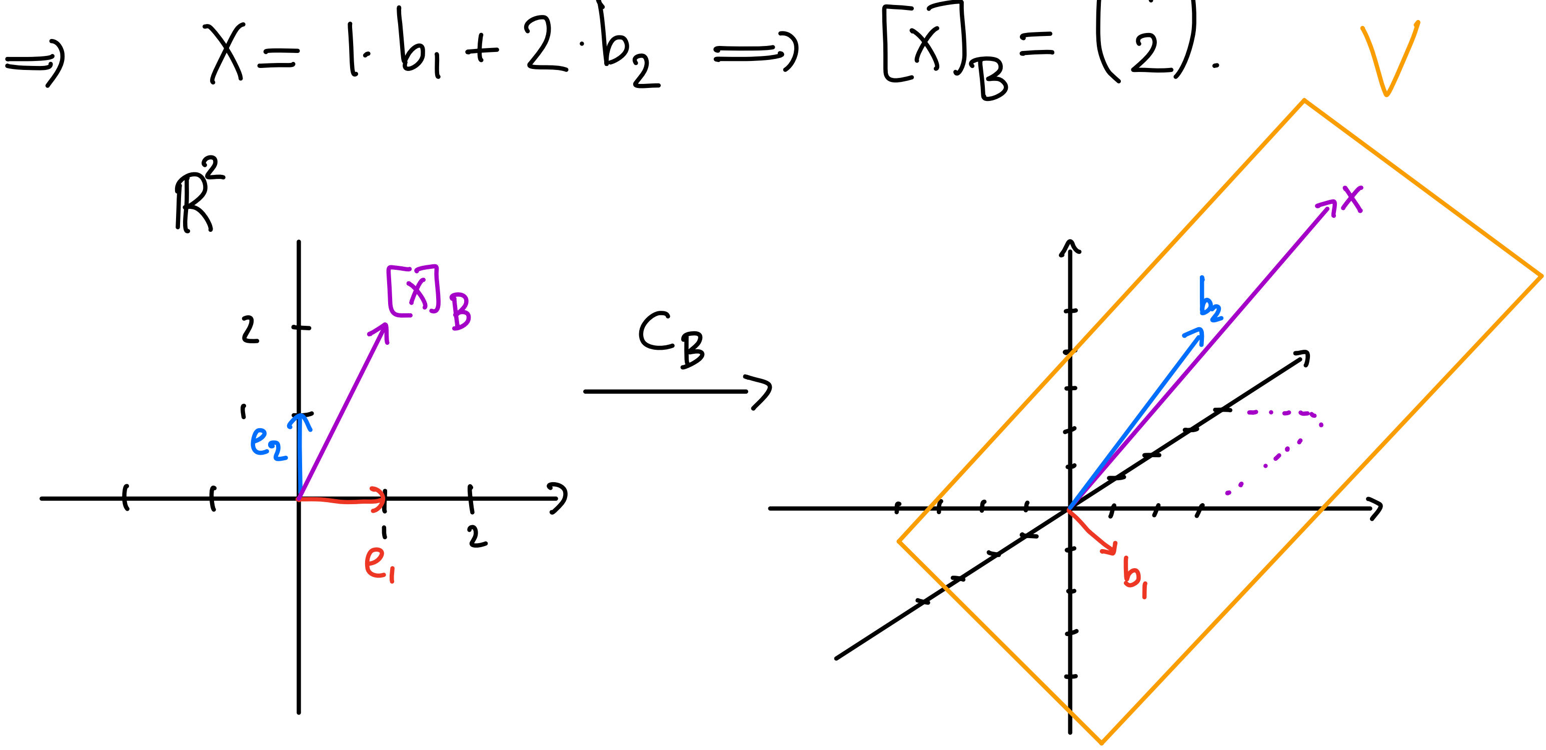
Is $x = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \in V$? What is $[x]_{\mathcal{B}}$?

We need to solve $\lambda_1 b_1 + \lambda_2 b_2 = x$.

$$\begin{array}{c} \begin{array}{ccc} b_1 & b_2 & x \end{array} \\ \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 2 & 4 \\ -1 & 3 & 5 \end{array}\right) \end{array} \sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 2 & 4 \\ 0 & 4 & 8 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right).$$

$\Rightarrow \quad x = 1 \cdot b_1 + 2 \cdot b_2 \Rightarrow [x]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$



Proposition 11.4 Let $\mathcal{B}=(b_1,\dots,b_m)$ be a basis of $V\subset\mathbb{R}^n$, $x,y\in V$, $\mu\in\mathbb{R}$, then

i) $[x+y]_{\mathcal{B}}=[x]_{\mathcal{B}}+[y]_{\mathcal{B}}$,

ii) $[\mu x]_{\mathcal{B}}=\mu[x]_{\mathcal{B}}$,

iii) $[0]_{\mathcal{B}}=0$.

Proof: $[x]_{\mathcal{B}}=C_{\mathcal{B}}^{-1}(x)$. $C_{\mathcal{B}}$ is linear $\Rightarrow C_{\mathcal{B}}^{-1}$ linear.

All properties follow from the linearity of $C_{\mathcal{B}}^{-1}$. $\square$

Definition 11.5 Let $\mathcal{B}=(b_1,\dots,b_n)$ be a basis of $\mathbb{R}^n$. The <u>change-of-basis matrix</u> associated with $\mathcal{B}$ is

$$S_{\mathcal{B}}=[C_{\mathcal{B}}]=\begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix}\in\mathbb{R}^{n\times n}.$$

Remark:
- $S_{\mathcal{B}}$ is invertible.
- For all $x\in\mathbb{R}^n$: $S_{\mathcal{B}}[x]_{\mathcal{B}}=C_{\mathcal{B}}([x]_{\mathcal{B}})$
$$=C_{\mathcal{B}}(C_{\mathcal{B}}^{-1}(x))=x.$$

Definition 11.6 Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, B_1 basis of $\mathbb{R}^n$, B_2 basis of $\mathbb{R}^m$.

The <u>matrix of F with respect to B_1 and B_2</u> is

$$[F]_{B_1}^{B_2} = \left[C_{B_2}^{-1} \circ F \circ C_{B_1} \right].$$

In the case $n=m$ and $B_1 = B_2$ we write $[F]_{B_1} = [F]_{B_1}^{B_1}$.

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "F"] & \mathbb{R}^m \arrow[d, shift left, "C_B^{-1}"] \\
\mathbb{R}^n \arrow[u, "C_{B_1}"] \arrow[r, "C_B^{-1} \circ F \circ C_{B_1}"'] & \mathbb{R}^m \arrow[u, shift left, "C_{B_2}"]
\end{tikzcd}$$

Proposition 11.7 Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, B_1 basis of $\mathbb{R}^n$, B_2 basis of $\mathbb{R}^m$.

i) We have $[F]_{B_1}^{B_2} = S_{B_2}^{-1} [F] S_{B_1}$.

ii) If $B_1 = (b_1, \dots, b_n)$ then

$$[F]_{B_1}^{B_2} = \begin{pmatrix} | & & | \\ [F(b_1)]_{B_2} & \cdots & [F(b_n)]_{B_2} \\ | & & | \end{pmatrix}.$$

Proof: i) $[F]^{B_2}_{B_1} = [C_{B_2}^{-1} \circ F \circ C_{B_1}] = [C_{B_2}^{-1}][F][C_{B_1}]$
$$= S_{B_2}^{-1}[F]S_{B_1}.$$

ii) The i-th column of $[F]^{B_2}_{B_1}$ is
$$[F]^{B_2}_{B_1} e_i = C_{B_2}^{-1}(F(C_{B_1}(e_i))) = C_{B_2}^{-1}(F(b_i))$$
$$= [F(b_i)]_{B_2}. \quad \square$$

Example 38

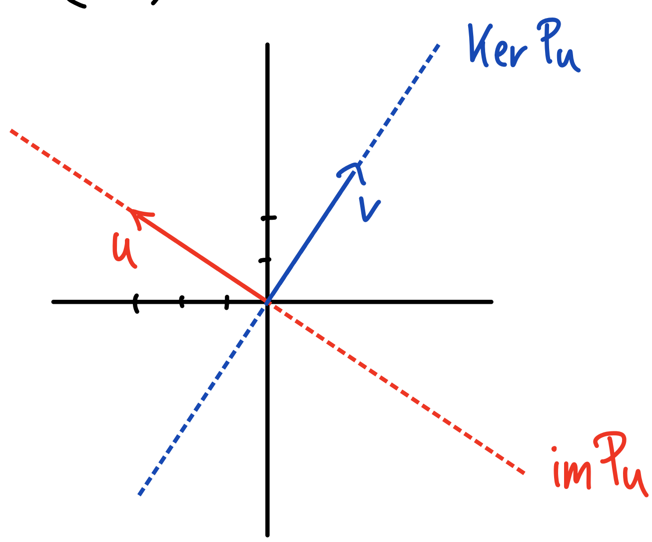
1) $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $B_1 = (e_1, \dots, e_n)$ $\quad [C_{B_1}] = I_n$

$B_2 = (e_1, \dots, e_m)$ $\quad [C_{B_2}] = I_m$

Then $[F]^{B_2}_{B_1} = [F]$

2) $P_u: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $u = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

$$\operatorname{Ker} P_u = \{x \in \mathbb{R}^2 \mid u \cdot x = 0\}$$
$$= \operatorname{span}\Big\{\underbrace{\begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{= v}\Big\}$$



$\operatorname{im} P_u = \operatorname{span}\{u\}$.

u and v are linearly indep. $\Rightarrow$ $\mathcal{B} = (u, v)$ is a basis of $\mathbb{R}^2$.

We have $P_u(u) = u$ and $P_u(v) = 0$, i.e.

$$[P_u]_{\mathcal{B}} = [P_u]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} | & | \\ [P_u(u)]_{\mathcal{B}} & [P_u(v)]_{\mathcal{B}} \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Change-of-basis matrix:

$$S_{\mathcal{B}} = \begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 2 & 3 \end{pmatrix}.$$

Inverse: $$S_{\mathcal{B}}^{-1} = \begin{pmatrix} -\frac{3}{13} & \frac{2}{13} \\ \frac{2}{13} & \frac{3}{13} \end{pmatrix}$$ (Check!)

By Prop 11.7 we have $[P_u]_{\mathcal{B}} = [P_u]_{\mathcal{B}}^{\mathcal{B}} = S_{\mathcal{B}}^{-1} [P_u] S_{\mathcal{B}}$

$$\Rightarrow [F] = S_{\mathcal{B}} [P_u]_{\mathcal{B}} S_{\mathcal{B}}^{-1} = \begin{pmatrix} -3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{3}{13} & \frac{2}{13} \\ \frac{2}{13} & \frac{3}{13} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{9}{13} & -\frac{6}{13} \\ -\frac{6}{13} & \frac{4}{13} \end{pmatrix}.$$

You can check that this gives the same matrix you calculated in HW4 Ex 1 (ii).

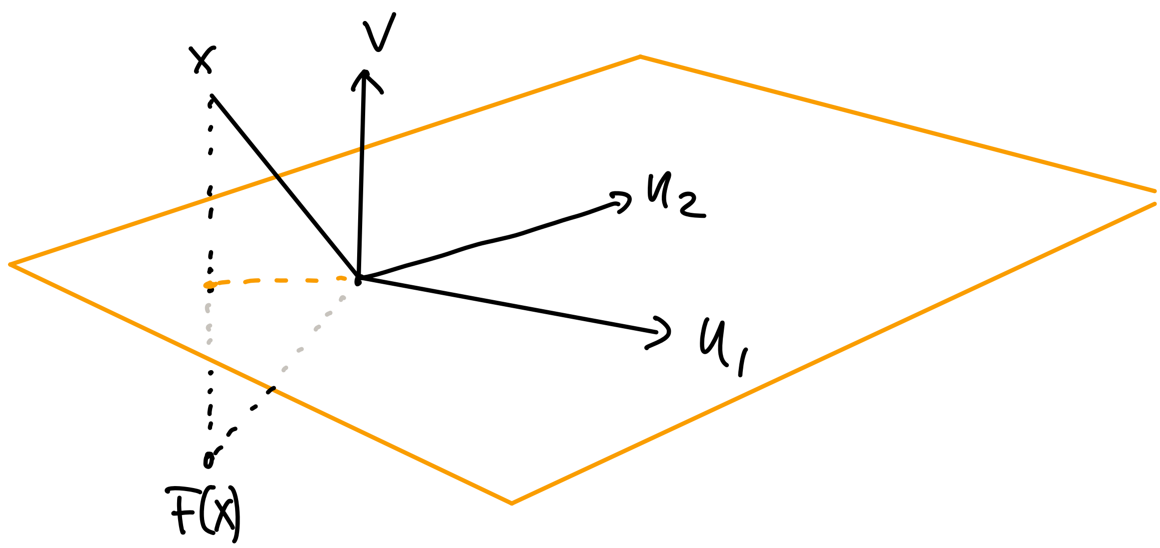
3) Let $F: \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection in the plane $U = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}_{u_2} \right\}$.

Goal: Determine the matrix of $[F]$.

Idea: • Find "good" basis B, where we can write down $[F]_B$ directly.

• For this try to find $v \in \mathbb{R}^3$ which is orthogonal to u_1, u_2 and set $B = (u_1, u_2, v)$.

(Notice: u_1, u_2 are lin. indep.)



To find $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3$ with $v \cdot u_1 = v \cdot u_2 = 0$ we need to solve $\begin{cases} v_1 + v_2 = 0 \\ v_2 + v_3 = 0 \end{cases}$.

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$\Rightarrow$ all solutions are given by $t \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.
We choose $t=1$, i.e. set

$$v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Check: u_1, u_2, v are linearly independent

$\Rightarrow$ $B = (u_1, u_2, v)$ is a basis of $\mathbb{R}^3$.

We have
$$F(u_1) = u_1 \Rightarrow [F(u_1)]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$F(u_2) = u_2 \Rightarrow [F(u_2)]_B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$F(v) = -v \Rightarrow [F(v)]_B = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow [F]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Change-of-basis matrix:

$$S_B = \begin{pmatrix} | & | & | \\ u_1 & u_2 & v \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

Check: $S_B^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 & -1 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$

$$\Rightarrow \quad [F] = S_B \, [F]_B \, S_B^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & 1 & -1 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \underline{\underline{\frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}}}.$$