Linear Algebra I — Fall 2023

Recall

Definition 9.2 Vectors $v_1, \dots, v_\ell \in \mathbb{R}^m$ are called <u>linearly independent</u> if the equation

$$\lambda_1 v_1 + \dots + \lambda_\ell v_\ell = 0 \qquad (\lambda_1, \dots, \lambda_\ell \in \mathbb{R})$$

just has the <u>unique</u> solution $\lambda_1 = \lambda_2 = \dots = \lambda_\ell = 0$.

Otherwise $v_1, \dots, v_\ell$ are called <u>linearly dependent</u>.

Theorem 9.3 Let $v_1, \dots, v_\ell \in \mathbb{R}^m$. The following statements are equivalent:

i) $v_1, \dots, v_\ell$ are linearly dependent.

iii) There exists a $j = 1, \dots, \ell$ with

$$\operatorname{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_\ell\} = \operatorname{span}\{v_1, \dots, v_\ell\}.$$

Lemma 9.4 $V \subset \mathbb{R}^n$ subspace, $v_1, \dots, v_\ell \in V$ lin. indep.
If $V = \operatorname{span}\{w_1, \dots, w_m\}$ for some $w_1, \dots, w_m \in \mathbb{R}^n \Rightarrow \ell \le m$.

Lemma 9.5 If $v_1, \dots, v_\ell \in \mathbb{R}^n$ are linearly independent and $w \in \mathbb{R}^n$ with $w \notin \operatorname{span}\{v_1, \dots, v_\ell\}$ then $v_1, \dots, v_\ell, w$ are linearly independent.

Proof: Assume that $\lambda_1 v_1 + \dots + \lambda_\ell v_\ell + \mu w = 0$. $\quad (\lambda_1, \dots, \lambda_\ell, \mu \in \mathbb{R})$

If $\mu \neq 0$, then $w = \left(\frac{\lambda_1}{\mu}\right) v_1 + \dots + \left(\frac{\lambda_\ell}{\mu}\right) v_\ell \in \operatorname{span}\{v_1, \dots, v_\ell\}$.

Hence $\mu = 0 \Rightarrow \lambda_1 v_1 + \dots + \lambda_\ell v_\ell = 0$

$$\Rightarrow \lambda_1 = \dots = \lambda_\ell = 0$$

$v_1, \dots, v_\ell$ are lin. indep.

$$\Rightarrow v_1, \dots, v_\ell, w \text{ are lin. indep.}$$

$\square$

§ 10 Bases & Dimension

We saw that if $v_1, \dots, v_\ell \in \mathbb{R}^n$ are linearly dependent, then there exist a $1 \le j \le \ell$ such that

$$\operatorname{span}\{v_1, \dots, v_\ell\} = \operatorname{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_\ell\}.$$

Therefore we will be just interested in the case when $v_1, \dots, v_\ell$ are linearly independent.

Definition 10.1 Let $V \subset \mathbb{R}^n$ be a subspace.

Vectors $v_1, \dots, v_\ell \in V$ form a <u>basis of V</u> if

i) $V = \operatorname{span}\{v_1, \dots, v_\ell\}$

ii) $v_1, \dots, v_\ell$ are linearly independent

We set $\operatorname{span} \emptyset := \{0\}$, so $\emptyset = \{\}$ is a basis of $\{0\}$.

Notation: We say $\{v_1, \dots, v_\ell\}$ is a basis of V.

Later we consider <u>ordered</u> basis and write $(v_1, \dots, v_\ell)$

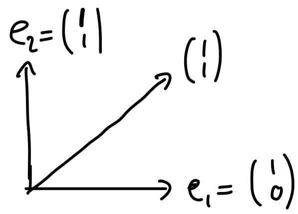
order matters

Note: $\{1,2\} = \{2,1\}$ but $(1,2) \neq (2,1)$

Example 34

1)

$\{e_1, e_2\}$, $\{e_1, \binom{1}{1}\}$, $\{e_2, \binom{1}{1}\}$ are three different bases of $\mathbb{R}^2$.



2) $U = \text{span}\left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix}} \right\}$.

Find a basis for U.

We have seen in Example 33:

- v_1, v_2, v_3 are linearly dependent because $-2v_1 + 3v_2 + v_3 = 0$.
- $v_3 \in \text{span}\{v_1, v_2\}$

$\overset{}{\underset{\text{Lemma 9.1}}{\Longrightarrow}}$ $\underline{U = \text{span}\{v_1, v_2\}}$

- $\underline{v_1, v_2 \text{ are linearly independent}}$

$\Longrightarrow$ $\{v_1, v_2\}$ is a basis of U.

3) For any $n \geq 1$ $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{R}^n$.
This basis is called the <u>standard basis</u>.

Theorem 10.2 Let $V \subset \mathbb{R}^n$ be a subspace

i) V has a basis.

ii) All bases of V have the same number of elements.

iii) If $v_1, \dots, v_\ell \in V$ are linearly independent then there exist $u_{\ell+1}, \dots, u_t \in V$ such that $\{v_1, \dots, v_\ell, u_{\ell+1}, \dots, u_t\}$ is a basis of V. (possibly no u are needed!)

iv) If $V = \operatorname{span}\{w_1, \dots, w_m\}$ then there exists a subset $\{u_1, \dots, u_\ell\} \subseteq \{w_1, \dots, w_m\}$ such that $\{u_1, \dots, u_\ell\}$ is a basis of V. equality possible!

Proof:

ii): Let $\{v_1, \dots, v_\ell\}$ and $\{w_1, \dots, w_m\}$ be a basis of V

$$\left.\begin{array}{l} v_1, \dots, v_\ell \in V \text{ are lin. indep}, \ \operatorname{span}\{w_1, \dots, w_m\} = V \overset{\text{Lemma 9.4}}{\Longrightarrow} \ell \le m \\ w_1, \dots, w_m \in V \text{ are lin. indep}, \ \operatorname{span}\{v_1, \dots, v_\ell\} = V \overset{\text{Lemma 9.4}}{\Longrightarrow} m \le \ell \end{array}\right\} \ell = m.$$

iv): ① If $\{w_1, \dots, w_m\}$ are lin. indep. we are done and $\{w_1, \dots, w_m\}$ is a basis of V. (The subset is just everything)

② Assume $\{w_1, \dots, w_m\}$ are lin. dependent.

By Theorem 9.3 there exist a $j = 1, \dots, m$ with

$$\operatorname{span}\{w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_m\} = \operatorname{span}\{w_1, \dots, w_m\} = V$$

Now repeat ① & ② with $\{w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_m\}$, i.e. remove vectors w_j until the remaining vectors are lin. indep.

iii) Assume $v_1, \dots, v_\ell \in V$ are lin. indep.

① If $\operatorname{span}\{v_1, \dots, v_\ell\} = V$ we are done and $\{v_1, \dots, v_\ell\}$ is a basis of V.

② If $\operatorname{span}\{v_1, \dots, v_\ell\} \neq V$ then there exists $u \in V$ with $u \notin \operatorname{span}\{v_1, \dots, v_\ell\}$. Set $u_{\ell+1} = u$. By Lemma 9.5 $v_1, \dots, v_\ell, u_{\ell+1}$ are lin. indep.

Repeat ① & ② until $V = \operatorname{span}\{v_1, \dots, v_\ell, u_{\ell+1}, \dots, u_t\}$.

i) If $V = \{0\}$ then $\emptyset = \{\}$ is a basis. Else construct a basis using iii).

$(\operatorname{span} \emptyset := \{0\})$

Definition

Definition 10.3 Let $V \subset \mathbb{R}^n$ be a subspace.

The <u>dimension of V</u>, denoted by $\dim(V)$, is the number of elements in a basis of V.

(Notice: this definition makes sense because of Thm 10.2 i) & ii))

Example 35 1) $\dim \mathbb{R}^n = n$, because $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{R}^n$.

2) The dimension of $U = \operatorname{span}\left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}, \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix} \right\}$ is $\dim(U) = 2$ because $\{v_1, v_2\}$ is a basis.

Corollary 10.4 $V \subset \mathbb{R}^n$ subspace with $\dim V = m$ and $v_1, \dots, v_m \in V$. Then the following statements are equivalent.

i) $v_1, \dots, v_m$ are lin. indep.

ii) $V = \operatorname{span}\{v_1, \dots, v_m\}$.

iii) $\{v_1, \dots, v_m\}$ is a basis of V.

Proof: i) $\Rightarrow$ ii): If $V \neq \operatorname{span}\{v_1, \dots, v_m\}$ then by Thm. 10.2 iii) there would exist a basis with more than m elements. But $\dim(V) = m$ i.e. $V = \operatorname{span}\{v_1, \dots, v_m\}$.

ii) $\Rightarrow$ i): If $v_1, \dots, v_m$ would be lin. dependent then by Thm. 9.3. there would exist a basis with less than m elements. But $\dim(V) = m$.

$\Rightarrow$ $v_1, \dots, v_m$ are lin. independent.

i) + ii) $\Leftrightarrow$ iii) by definition. $\square$

Example 36 Determine bases for $\operatorname{Ker}(F)$ and $\operatorname{im}(F)$ of the following linear map:

$$F: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

$$x \longmapsto \begin{pmatrix} 1 & -2 & 1 & 2 \\ 2 & -4 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix} x$$

Kernel:

$$[F] = \begin{pmatrix} 1 & -2 & 1 & 2 \\ 2 & -4 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 1 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{rref}([F])$$

$x \in \mathrm{Ker}(F) \iff [F]x = 0 \iff$

$$\begin{aligned} x_1 &= 2t_1 - t_2 \\ x_2 &= t_1 \\ x_3 &= -t_2 \\ x_4 &= t_2 \end{aligned} \qquad t_1, t_2 \in \mathbb{R}$$

$$\iff x = t_1 \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{= v_1} + t_2 \underbrace{\begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{= v_2} \implies \mathrm{Ker}(F) = \mathrm{span}\{v_1, v_2\}.$$

v_1, v_2 lin. indep. ?

$$0 = t_1 v_1 + t_2 v_2 = \begin{pmatrix} 2t_1 - t_2 \\ t_1 \\ -t_2 \\ t_2 \end{pmatrix} \implies t_1 = t_2 = 0$$

$\implies v_1, v_2$ lin. indep

$\implies \{v_1, v_2\}$ basis of $\mathrm{Ker}(F)$.

Image:

$$\operatorname{im}(F) = \operatorname{span}\left\{ \underset{u_1}{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}, \underset{u_2}{\begin{pmatrix} -2 \\ -4 \\ 0 \end{pmatrix}}, \underset{u_3}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, \underset{u_4}{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}} \right\}$$

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \lambda_4 u_4 = 0 \iff [F] \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = 0$$

$$\iff \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \in \operatorname{Ker}(F)$$

Saw before

$$\Rightarrow \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = t_1 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \quad \text{for } t_1, t_2 \in \mathbb{R}.$$

$t_1 = 1,\ t_2 = 0$: $\quad 2u_1 + u_2 = 0 \Rightarrow u_2 = -2u_1$
$\in \operatorname{span}\{u_1, u_3\}$

$t_1 = 0,\ t_2 = 1$: $\quad -u_1 - u_3 + u_4 = 0 \Rightarrow u_4 = u_1 + u_3$
$\in \operatorname{span}\{u_1, u_3\}$

$$\Rightarrow \operatorname{im}(F) = \operatorname{span}\{u_1, u_3\}$$

u_1 and u_3 are lin. indep because $\lambda_1 u_1 + \lambda_3 u_3 = 0$:

$$\begin{array}{c} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right) \\ \begin{array}{cc} u_1 & u_3 \end{array} \end{array} \sim \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \lambda_1 = \lambda_3 = 0.$$

$\Rightarrow \{u_1, u_3\}$ is a basis of $\mathrm{im}(F)$.

General calculation of a basis for $\mathrm{Ker}(F)$ and $\mathrm{im}(F)$:

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear map.
$x \longmapsto Ax$

- Let $\mathrm{rref}(A)$ have pivot elements in columns $c_1, \dots, c_r$.

  Then the columns $c_1, \dots, c_r$ in A form a basis of $\mathrm{im}(F)$.

  $\Rightarrow \quad \dim(\mathrm{im}(F)) = \#$ pivot elements in $\mathrm{rref}(A)$

- The vectors obtained in the "standard parametrisation" (i.e. for each free variable there is a parameter t_i) of the solutions of $F(x) = 0$ form a basis of $\mathrm{Ker}(F)$.

  $\Rightarrow \quad \dim(\mathrm{Ker}(F)) = \#$ free variables

As a consequence we get the following

Theorem 10.5 For a linear map $F: \mathbb{R}^n \to \mathbb{R}^m$ we have

$$\dim(\ker(F)) + \dim(\operatorname{im}(F)) = n.$$

Proof: $n = \#$ columns of $[F]$

$\dim(\ker(F)) = \#$ columns without pivot elements.

$\dim(\operatorname{im}(F)) = \#$ columns with pivot elements.