Linear Algebra I

§0 Introduction "What is linear algebra?"

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線形代数学

line shape change number study

linear math

algebra

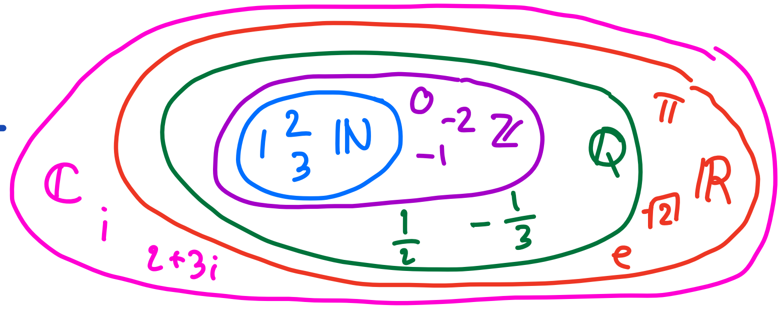
Linear Algebra = "study of solutions of linear equations."

More precisely: Study of "flat" spaces, called vector spaces / linear spaces.
(e.g. lines, planes, 3dim-space, ...)
and maps between these spaces, called linear maps (e.g. rotations, projections, ...).

In this course we will just consider $\mathbb{R}^n$.
(more general: Linear Algebra II)

- Earliest origins in China more than 2000 years ago.
- Modern form: early 20^{th} century.
- Universally useful in Mathematics, Sciences & Engineering (e.g. 3D Graphics) and life.

§ 1 Linear systems



$\mathbb{R}$: The set of real numbers, e.g. $1, 2, -1, \frac{1}{3}, \sqrt{2}, \pi, e, \dots$.

"$b \in \mathbb{R}$": b is an element in $\mathbb{R}$, i.e b is a real number.

Definition 1.1

i) **Linear equation** (lin. eq): $a_1x_1 + \dots + a_nx_n = b$ $a_1, \dots, a_n, b \in \mathbb{R}$

ii) **Linear system**: A finite collection of lin. eq..

iii) **Solution of a linear system**: Simultaneous solution to all of its eq..

Goal: Given a linear system find all of its solutions.

Example 2

We use the numbering of the lecture notes.

"This implies"

$$\text{②}\ \begin{cases} x_1 + 3x_2 = 1 \\ -2x_1 + x_2 = 2 \end{cases} \Longrightarrow \left(\tfrac{1}{7}\right) \begin{cases} x_1 + 3x_2 = 1 \\ 7x_2 = 4 \end{cases}$$

"Add 2·first eq. to the second"

"multiply second eq. by $\frac{1}{7}$."

$$\Longrightarrow (-3) \begin{cases} x_1 + 3x_2 = 1 \\ x_2 = \frac{4}{7} \end{cases}$$

$$\Longrightarrow \begin{cases} x_1 = 1 - \overbrace{3 \cdot \tfrac{4}{7}}^{\frac{12}{7}} = -\frac{5}{7} \\ x_2 = \frac{4}{7} \end{cases}$$

Is $x_1 = -\frac{5}{4},\ x_2 = \frac{4}{7}$ a solution to the original linear system?

Yes, because the operations work also in reverse.

$$3\ \begin{cases} x_1 = -\frac{5}{7} \\ x_2 = \frac{4}{7} \end{cases} \Rightarrow 7\ \begin{cases} x_1 + 3x_2 = 1 \\ x_2 = \frac{4}{7} \end{cases} \Rightarrow -2\ \begin{cases} x_1 + 3x_2 = 1 \\ 7x_2 = 4 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 + 3x_2 = 1 \\ -2x_1 + x_2 = 2 \end{cases}$$

Therefore:

$$\begin{cases} x_1 + 3x_2 = 1 \\ -2x + x_2 = 2 \end{cases} \iff \begin{cases} x_1 = -\frac{5}{7} \\ x_2 = \frac{4}{7} \end{cases}$$

Iff
"if and only if".

This linear system has exactly one solution.

Example 3

Now consider the following linear system

-2   -3

$$\begin{cases} x_1 - 9x_2 - 3x_3 + x_4 = 4 \\ 3x_1 - 2x_2 + x_3 - 2x_4 = 2 \\ 2x_1 + 7x_2 + 4x_3 - 3x_4 = -2 \\ 25x_2 + 9x_3 - 2x_4 = -4 \end{cases} \iff \begin{cases} x_1 - 9x_2 - 3x_3 + x_4 = 4 \\ 25x_2 + 10x_3 - 5x_4 = -10 \\ 25x_2 + 10x_3 - 5x_4 = -10 \\ 25x_2 + 9x_3 - 2x_4 = -4 \end{cases}$$

-1   -1

$\iff$

-3   10

$$\begin{cases} x_1 - 9x_2 - 3x_3 + x_4 = 4 \\ 25x_2 + 10x_3 - 5x_4 = -10 \\ 0 = 0 \quad \text{(We can ignore this eq.)} \\ -x_3 + 3x_4 = 6 \end{cases} \iff \begin{array}{l} \\ \frac{1}{25} \\ -1 \end{array} \begin{cases} x_1 - 9x_2 - 8x_4 = -14 \\ 25x_2 + 25x_4 = 50 \\ -x_3 + 3x_4 = 6 \end{cases}$$

row-reduced echelon form

9

$$\iff \begin{cases} x_1 - 9x_2 - 8x_4 = -14 \\ x_2 + x_4 = 2 \\ x_3 - 3x_4 = -6 \end{cases} \iff \boxed{\begin{cases} x_1 + x_4 = 4 \\ x_2 + x_4 = 2 \\ x_3 - 3x_4 = -6 \end{cases}} \quad (*)$$

$$\iff \begin{cases} x_1 = 4 - x_4 \\ x_2 = 2 - x_4 \\ x_3 = -6 + 3x_4 \end{cases}$$

The variable x_4 can be choosen arbitrary.

We set $x_4 = t$ for some $t \in \mathbb{R}$.

All the solutions are given by

$$\begin{aligned} x_1 &= 4 - t \\ x_2 &= 2 - t \\ x_3 &= -6 + 3t \\ x_4 &= t \end{aligned} \quad \text{for } t \in \mathbb{R}.$$

Equation (*): Each equation contains a variable that occurs in no other equation: (x_1, x_2, x_3), called <u>pivot variables</u>.

The other variables (x_4) are called <u>free variables</u>.

This equation is in <u>row-reduced echelon form</u> (see HW1 for a definition).

Later we will define this for matrices.

How to get to this form?

Definition 1.2 <u>Elementary row operations</u>

(R1) Add a multiple of an eq. to another.

(R2) Multiply an eq. with a <u>non-zero</u> number.

(R3) Change the order of the eq.

Since all elementary row operations work in reverse we get:

Proposition 1.3 Applying an elementary row operation to a linear system does not change the set of all solutions of this linear system.

Algorithm 1.4

General solutions method

aka "row reduction" (Gaussian elimination)

(Brings a linear system to row-reduced echelon form)

I: Downwards:

1) Make the first eq. contain the first variable (R3)
2) Make the coefficient of this variable = 1 (R2)
3) Eliminate this variable from all other eq. (R1)
4) Iterate with first occuring variable in the remaining eq.

II: Upwards

1) Let x_i be the first variable in the last eq. Eliminate x_i from all other eq. (R1)

2) Go to previous eq. and iterate.

Example 4

(R3) $\quad$ (R1)

$$\begin{cases} \quad\quad\quad\quad\; -x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 - x_4 = 0 \\ x_1 + x_2 - x_3 \quad\quad\; = 1 \end{cases} \iff \begin{cases} x_1 + x_2 + x_3 - x_4 = 0 \\ \quad\quad\quad\; -x_3 + x_4 = 0 \\ x_1 + x_2 - x_3 \quad\quad\; = 1 \end{cases}$$

-1

$\Leftrightarrow$

(R2) $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ (R1)

$$\begin{cases} x_1 + x_2 + x_3 - x_4 = 0 \\ \quad\quad -x_3 + x_4 = 0 \quad \ominus 1 \\ \quad\quad -2x_3 + x_4 = 1 \end{cases} \Leftrightarrow \begin{cases} x_1 + x_2 + x_3 - x_4 = 0 \\ \quad\quad x_3 - x_4 = 0 \quad (2) \\ \quad\quad -2x_3 + x_4 = 1 \end{cases}$$

$\Leftrightarrow$

(R2) $\quad\quad\quad\quad\quad\quad\quad\quad\quad$ From here on upwards (R1)

$$\begin{cases} x_1 + x_2 + x_3 - x_4 = 0 \\ \quad\quad x_3 - x_4 = 0 \\ \quad\quad -x_4 = 1 \quad \ominus 1 \end{cases} \Leftrightarrow \begin{cases} x_1 + x_2 + x_3 - x_4 = 0 \\ \quad\quad x_3 - x_4 = 0 \\ \quad\quad x_4 = -1 \quad (1)\ (1) \end{cases}$$

(R1)

$$\Leftrightarrow \begin{cases} x_1 + x_2 + x_3 = -1 \\ \quad\quad x_3 = -1 \quad \ominus 1 \\ \quad\quad x_4 = -1 \end{cases} \Leftrightarrow \begin{cases} x_1 + x_2 = 0 \\ \quad\quad x_3 = -1 \\ \quad\quad x_4 = -1 \end{cases}$$

Pivot variables

free variable

The free variables can be choosen arbitrary.

We set $x_2 = t$, $t \in \mathbb{R}$.

All solutions are given by
$$\begin{aligned} x_1 &= -t \\ x_2 &= t \\ x_3 &= -1 \\ x_4 &= -1 \end{aligned} \quad \text{for } t \in \mathbb{R}.$$