Homework 3: Functions & Linear maps

Deadline: 12th November, 2023

Exercise 1. (3+3+4=10 Points) We define the following four functions:

$$f_1 : \mathbb{R} \longrightarrow \mathbb{R}^2, \quad x \longmapsto \begin{pmatrix} 1-\cos(x) \\ \sin(x) \end{pmatrix}, \qquad f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_1 - x_2 \\ x_1 x_2 \end{pmatrix},$$

$$f_3 : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto \frac{4x}{x^2+4}, \qquad f_4 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 3x_1 - x_2 \\ 2x_1 - x_2 \\ x_1 - x_2 \end{pmatrix}.$$

(i) Calculate the image of each function, i.e. describe $\operatorname{im}(f_j)$ for $j = 1,2,3,4$ as explicit as possible. If you can not find a mathematical description try to describe the elements of the image in words.

(ii) Decide for each function if it is injective and/or surjective and/or bijective.

(iii) Decide which of the above functions are linear maps.

Justify your answers.

Exercise 2. (5 Points) Show that there exist a unique linear map $T : \mathbb{R}^2 \to \mathbb{R}^3$ with the property

$$T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

What is the value of $T(x)$ for an arbitrary $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$? Determine the matrix of T.

Exercise 3. (2+2+3=7 Points)

(i) Let X be a finite set. Show that a function $f : X \to X$ is injective if and only if it is surjective.

(ii) Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map. Show that F can not be surjective.

(iii) Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Show that F is injective if and only if the only solution to $F(x) = 0$ is $x = 0$.

;

くま先生の 簡単数学用語 解説コーナー

Hello ~ クマ先生 here. With another homework comes another Japanese lesson~

This week, I've prepared five words: two nouns and three adjectives. First, the nouns:

集合（しゅうごう）　　写像（しゃぞう）

The two words are shuugou (**set**) and shazou (**map**, as in **linear map** (線形写像（せんけいしゃぞう）)). This time, both words are used almost exclusively used in mathematics. Next, we have three words to describe a function:

単射（たんしゃ）　　全射（ぜんしゃ）　　全単射（ぜんたんしゃ）

These three words are: tansha (**injective**), zensha (**surjective**), and zentansha (**bijective**). To describe a function (in Japanese), we combine the words. For example, injectiive function is 単射写像（たんしゃしゃぞう）.

Anyway, now, a breakdown of the individual 漢字（かんじ） that makes up these words:

集（しゅう） - This kanji means "**(to) gather**". It refers to how sets are gatherings of things. Common uses of this kanji is 集（あつ）まる (to gather / collect) and 集中（しゅうちゅう） (focus).

合（ごう） - This kanji means "**(to) match**". In a sense, sets are comprised of things (that are assumed to be similar). Everyday words that include this kanji include 間（ま）に合（あ）う (to make (it) in time) and 合（ごう）コン (Matchmaking Party).

写（しゃ） - This kanji means "**(to) copy**". Perhaps, it is used in the word for "map" since it "copies" one set onto another (by associating one element in the domain to one in the codomain). A common use of this kanji is 写真（しゃしん） (photo).

像（ぞう） - This kanji means "**image**". Alternatively, it also means **statue**, **picture** or **likeness (of)**. One word that uses this kanji is 仏像（ぶつぞう） (Buddha Statue)

射（しゃ） - This kanji means "**(to) shoot**" or "**(to) project**". This refers to how functions "shoot" elements of one set to elements of another set. This kanji is not common in everyday life.

単（たん） - This kanji means "**single**. It refers to how in an injective function, every element in the range are "shot by" (or associated with) exactly one element in the domain. Everyday words that use this kanji include 単語（たんご） (vocabulary), 簡単（かんたん） (simple), 単位（たんい） (credits), and 単価（たんか） (unit price).

全（ぜん） - This kanji means "**all**". It refers to how in a surjective function, every element in the codomain are associated with ("shot by") at least one element in the domain. This kanji has multiple readings (全（まった）く and 全（すべ）て), and almost all of them have the same meaning ("every"). Common words that use this kanji include 完全（かんぜん） (perfect), 安全（あんぜん） (safe), and 全然（ぜんぜん） (not at all).

And that's it for today's (Mathematical) Japanese word(s). またね～