Homework 2: Matrices & Vectors

Deadline: 29th October, 23:55, 2023

Exercise 1. (2+2 = 4 Points) Show that for all $A \in \mathbb{R}^{m\times n}$, $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we have

(i) $A(x+y) = Ax + Ay$,

(ii) $A(\lambda x) = \lambda(Ax)$.

(Without using Proposition 2.4. from the lecture).

Let $A \in \mathbb{R}^{m\times n}$ be a matrix. The **rank** $\mathrm{rk}(A)$ of A is the number of pivot elements in $\mathrm{rref}(A)$.
For example,

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{rref}(A) \qquad ,$$

and therefore the rank of A is $\mathrm{rk}(A) = 2$.

Exercise 2. (2+1+3 = 6 Points) Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ be a polynomial of degree 3 with real coefficients $a_0, a_1, a_2, a_3 \in \mathbb{R}$. For such a polynomial p define the vector v_p by

$$v_p = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^4 .$$

(i) Find a matrix $D \in \mathbb{R}^{4\times 4}$, such that $v_q = Dv_p$, where $q(x) = 2p(x) - p'(x)$.
(Here p' denotes the derivative of the polynomial p with respect to x).

(ii) Determine the rank of D in (i).

(iii) For an arbitrary polynomial p of degree 3, find a polynomial s, such that $v_p = Dv_s$.

Exercise 3. (2+2 = 4 Points) Let $A \in \mathbb{R}^{3\times 3}$ be a matrix.

(i) Show that if $\mathrm{rk}(A) = 3$ then there exists just one vector $x \in \mathbb{R}^3$ with $Ax = 0$.

(ii) Show that if $\mathrm{rk}(A) \leq 2$ then there exist infinitely many vectors $x \in \mathbb{R}^3$ with $Ax = 0$.

Exercise 4. (4+2 = 6 Points) Let $a, b, c, d \in \mathbb{R}$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(i) Show that $\mathrm{rk}(A) = 2$ if and only if $ad - bc \neq 0$.

(ii) We define the following subset of $\mathbb{R}^2$

$$L = \{x \in \mathbb{R}^2 \mid x = Av \text{ for some } v \in \mathbb{R}^2\}.$$

How does L look like if $\mathrm{rk}(A) = 1$? How does it look like if $\mathrm{rk}(A) = 2$?

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Hello ~ クマ先生 here. With another homework comes another Japanese lesson~

This week, I have three words prepared for this time:

行列（ぎょうれつ）　　方程式（ほうていしき）　　ベクトル

These three words are: gyouretsu (**matrix**), houteishiki (**equation**), and bekutoru (**vector**). While the Japanese word for vector is a direct transliteration of the English word vector, the other two are not so.

The first word (行列（ぎょうれつ）) also has a meaning in everyday life: a procession (e.g. a wedding procession).

The second word (方程式（ほうていしき）), however, appears more in the sciences. Famous examples include Schrödinger's Equation (シュレディンガー方程式（ほうていしき）) and Maxwell's Equations (マクスウェルの方程式（ほうていしき）). Generally, however, this word can be used to describe any type of equation. For example, 行列方程式（ぎょうれつほうていしき） (Matrix Equation), 代数方程式（だいすうほうていしき） (Algebraic Equation), ベクトル方程式（べくとるほうていしき） (Vector Equation) and 線型方程式（せんけいほうていしき） (Linear Equation).

Anyway, now, a breakdown of the individual 漢字（かんじ） that makes up these two words:

行（ぎょう） - This kanji means "**row(s)**". It refers to how matrices have "rows". In fact, the "row" of a matrix is called a 行（ぎょう） in Japanese. In everyday life, this kanji very common, as it is the kanji used in 行（い）きます (meaning "to go" in Japanese).

列（れつ） - This kanji means "**column(s)**". As such, the Japanese word for matrix literally means "rows of columns" or "columns of rows", which is an apt description of what a matrix is. In everyday life, this kanji shows up because of its other meaning : a queue (i.e. a queue in front of a shop).

方（ほう） - This kanji means "**direction**" or "**way**". It refers to how an equation gives direction(s) on how the variables are related to one another. This kanji appears in a few words in everyday life, including 方法（ほうほう） (method), (あの) 方（かた） (that person), and 貴方（あなた） (you; kanji literally means "precious person").

程（てい） - This kanji means "**about**". In a sense, 方程式（ほうていしき） (Equation) tells us something about some natural phenomenon. This kanji is uncommon in everyday life.

式（しき） - This kanji means "**formula**" or "**ceremony**". This kanji can be found in any ceremony in everyday life, such as 卒業式（そつぎょうしき） (Graduation Ceremony), 入学式（にゅうがくしき） (Entrance ceremony), and 結婚式（けっこんしき） (Wedding ceremony), among others.

And that's it for today's (Mathematical) Japanese word(s). またね ~