**LA1 - Final exam question session**

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**Linear Algebra I - Final exam** Fall 2022

Nagoya University, G30 Program Instructor: Henrik Bachmann

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$\begin{pmatrix}2&-4\\-1&2\end{pmatrix} \sim \begin{pmatrix}1&-2\\0&0\end{pmatrix}$

**1)** (12 Points) Let $A = \begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}$, $B = \begin{pmatrix}2&-4&1\\-1&2&-1\\1&-1&2\end{pmatrix}$ and $C = A^T B A$ $= 2$

i) Determine whether or not the matrices $A$, $B$, and $C$ are invertible and, if they are, compute their inverses.

$\left(B \,\middle|\, \begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix}\right) \sim \dots \sim \left(\begin{matrix}1&0&0\\0&1&0\\0&0&1\end{matrix} \,\middle|\, B^{-1}\right)$

ii) Determine $\mathrm{im}(C)$, $\ker(C)$ and $\mathrm{im}(C) \cap \ker(C)$.

iii) Give a basis for $\ker(C^n)$ for all $n \geq 1$.

$= \ker(C)$

$\sim \dots \sim \left(\begin{matrix}1&0&0\\0&1&0\\0&0&0\end{matrix} \,\middle|\, *\right)$

B not inv.

**2)** (14 Points) We define the subspace $U = \mathrm{span}\{u_1, u_2, u_3, u_4\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix}2\\2\\1\end{pmatrix}, \quad u_2 = \begin{pmatrix}-6\\-6\\-3\end{pmatrix}, \quad u_3 = \begin{pmatrix}-1\\3\\1\end{pmatrix}, \quad u_4 = \begin{pmatrix}6\\-2\\0\end{pmatrix}.$$

i) Determine a basis $B = (b_1, \dots, b_m)$ of $U$ and calculate its dimension.

ii) Calculate the coordinate vectors $[u_1]_B$, $[u_2]_B$,$[u_3]_B$ and $[u_4]_B$, where $B$ is the basis from i).

iii) Find a linear map $F : \mathbb{R}^3 \to \mathbb{R}^3$ with $\ker(F) = U^\perp$ and determine a basis of $\mathrm{im}(F)$.

**3)** (12 Points) Let $D \in \mathbb{R}^{2\times 2}$ be an arbitrary matrix. Which of the following sets are subspaces? Justify your answers.

i) $U_1 = \left\{ \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \in \mathbb{R}^3 \mid 2x_1 - x_2 = -x_3 + x_2 \right\}$.

ii) $U_2 = \left\{ x \in \mathbb{R}^{2023} \mid x \bullet x \geq -2023 \right\}$.

iii) $U_3 = \left\{ x \in \mathbb{R}^2 \mid \text{ There exists a } y \in \mathbb{R}^2 \text{ with } Dy = 3x \right\}$.

iv) $U_4 = \left\{ \begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{R}^2 \mid x_1 + x_2 = 0 \right\} \cup \left\{ \begin{pmatrix}x_1\\x_2\end{pmatrix} \in \mathbb{R}^2 \mid x_1 \cdot x_2 \leq 0 \right\}$.

**4)** (12 Points) We consider the vector $b = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$, the linear map

$$G : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$$
$$\begin{pmatrix}x_1\\x_2\end{pmatrix} \longmapsto \begin{pmatrix}1&1\\-1&1\\1&3\\-1&-1\end{pmatrix} \begin{pmatrix}x_1\\x_2\end{pmatrix},$$

$b_1 \quad b_2$

$(b_1, b_2)$ basis of $U$

and define the subspace $U = \mathrm{im}(G)$.

GSA

i) Show that $\dim(U) = 2$ and find an orthonormal basis $F = (f_1, f_2)$ of $U$.

ii) Determine the orthogonal projection $y = P_U(b)$ of $b$ onto $U$ and calculate $[y]_F$.

iii) Find a $x \in \mathbb{R}^2$ such that $\|G(x) - b\|$ is minimal.

**1)** (14 Points) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 1 & 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & -2 \\ -1 & 2 & 5 \end{pmatrix}$.

i) Determine whether or not the matrices $A$ and $B$ are invertible and, if they are, compute their inverses.

ii) Calculate the matrix $BA$ and decide if $BA^n$ is invertible for any integer $n \geq 1$.

iii) Determine if $\operatorname{im}(A) \cup \operatorname{im}(B)$ and $\operatorname{im}(A) \cap \operatorname{im}(B)$ are subspaces and, if they are, determine a basis.

**2)** (12 Points) We define the subspace $U = \operatorname{span}\{u_1, u_2, u_3, u_4\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}.$$

i) Determine a basis $B = (b_1, \dots, b_m)$ of $U$ and calculate its dimension.

ii) Calculate the coordinate vectors $[u_1]_B$, $[u_2]_B$,$[u_3]_B$ and $[u_4]_B$, where $B$ is the basis from i).

iii) Calculate an orthonormal basis $F = (f_1, \dots, f_m)$ for $U$ and determine $[u_1]_F$, $[u_2]_F$,$[u_3]_F$ and $[u_4]_F$

iv) Determine a basis for $U^\perp$.

**3)** (12 Points) Set $u = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and let $C \in \mathbb{R}^{2\times 2}$ be an arbitrary matrix. Which of the following subsets of $\mathbb{R}^2$ are subspaces? Justify your answers.

i) $U_1 = \left\{ x \in \mathbb{R}^2 \mid x \bullet x = x \bullet u \right\}$.

ii) $U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid 2x_1 - x_2 = 3x_2 \right\}$.

iii) $U_3 = \left\{ x \in \mathbb{R}^2 \mid Cx = x \right\}$.

iv) $U_4 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 \cdot x_2 \geq x_1 \right\}$.

**4)** (12 Points) Assume we have the following data points

| $i$ | 1 | 2 | 3 |
|---|---|---|---|
| $x_i$ | 1 | 2 | 3 |
| $y_i$ | 0 | $-1$ | $-3$ |

i) Find the line of best fit for the above data, i.e. find $m, n \in \mathbb{R}$ such that the function $l(x) = mx + n$ minimizes the sum of squares $\sum_{i=1}^{3} (l(x_i) - y_i)^2$.

ii) We define the following linear map

$$\begin{aligned} H : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\longmapsto \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

and set $V = \operatorname{im}(H)$. Determine a basis of $V$ and for $b = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix}$ calculate the orthogonal projection $P_V(b)$. Use your result to show that $b$ is not an element in $V$.

**After finishing this exam submit your solution as one pdf file at NUCT at the "Final exam" assignment.**

**Linear Algebra I - Final exam** Fall 2020
Nagoya University, G30 Program Instructor: Henrik Bachmann

**1)** (12 Points) Let $A = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ -2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

i) Compute the products $AB$ and $BA$, or explain why they are not defined.

ii) Determine whether or not the matrices $A$ and $B$ are invertible and, if they are, compute their inverses.

iii) Find all $x \in \mathbb{R}^3$ with $A^T A A^T A A^T A x = 0$. Justify you answer.

**2)** (12 Points) We define the subspace $U = \operatorname{span}\{u_1, u_2, u_3, u_4\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

i) Determine a basis $B = (b_1, \dots, b_m)$ of $U$ and calculate its dimension.

ii) Calculate the coordinate vectors $[u_1]_B$, $[u_2]_B$,$[u_3]_B$ and $[u_4]_B$, where $B$ is the basis from i).

iii) Find a linear map $G : \mathbb{R}^3 \to \mathbb{R}^3$ with $\operatorname{im}(G) = U$. What is the dimension of $\ker(G)$?

$\begin{pmatrix} | & | & 0 \\ u_1 & u_2 & 0 \\ | & | & 0 \end{pmatrix}$

**3)** (12 Points) Set $u = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Which of the following subsets of $\mathbb{R}^2$ are subspaces? Justify your answers.

i) $U_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid 2x_1 - 3x_2 = x_1 \right\}$.

ii) $U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 \text{ is an integer, i.e. } x_1 \in \{\dots, -2, -1, 0, 1, 2, \dots\} \right\}$.

iii) $U_3 = \left\{ x \in \mathbb{R}^2 \mid x \notin \operatorname{span}\{u\} \right\}$.

iv) $U_4 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \bullet u = x_1 \right\}$.

**4)** (14 Points) We define the following linear map

$$\begin{aligned} H : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\longmapsto \begin{pmatrix} 1 & 1 \\ -2 & 4 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

i) Show that $\dim(\operatorname{im}(H)) = 2$.

ii) Calculate an orthonormal basis $(f_1, f_2)$ for $\operatorname{im}(H)$.

iii) Find a vector $v \in \mathbb{R}^3$, such that $B = (f_1, f_2, v)$ is an orthonormal basis for $\mathbb{R}^3$.

iv) Calculate $[H(x)]_B$ for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$.

v) Find a $x \in \mathbb{R}^2$ such that $\|H(x) - b\|$ is minimal, where $b = \begin{pmatrix} -5 \\ 1 \\ -1 \end{pmatrix}$.

After finishing this exam please send your solution as **one pdf file** to
henrik.bachmann@math.nagoya-u.ac.jp

**Linear Algebra I - Final exam** Fall 2019

Nagoya University, G30 Program Instructor: Henrik Bachmann

$\mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$x \longmapsto Bx$

**1)** (12 Points) Let $A = \begin{pmatrix} 0 & 1 & -2 & 3 \\ 1 & -2 & 3 & -4 \\ -2 & 3 & -4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$.

$y \overset{?}{\in} \mathrm{im}(B)$

i) Compute the products $AB$ and $BA$, or explain why they are not defined.

ii) Determine whether or not the matrices $A$ and $B$ are invertible and, if they are, compute their inverses.

iii) Calculate $\mathrm{im}(B)$ and $\ker(B)$.

$Bx = y$

$x = B^{-1}y$

**2)** (14 Points) We define the subspace $U = \mathrm{span}\{u_1, u_2, u_3\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

i) Determine a basis $B = (b_1, \dots, b_m)$ of $U$ and calculate its dimension.

ii) Calculate the coordinate vectors $[u_1]_B$, $[u_2]_B$ and $[u_3]_B$, where $B$ is the basis you determined in i).

iii) Determine a basis for $U^\perp$.

iv) Find a linear map $G : \mathbb{R}^2 \to \mathbb{R}^3$ with $\ker(G) = \{0\}$ and $\mathrm{im}(G) = U$.

**3)** (10 Points) Which of the following subsets of $\mathbb{R}^2$ are subspaces? Justify your answers.

i) $U_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 + x_2 = x_1 x_2 \right\}$.

ii) $U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid 2x_1 = x_1 + x_2 \right\}$.

iii) $U_3 = \mathrm{span}\left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \bigcup \mathrm{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

(Friendly reminder: $\cup$ is the union of two sets)

**4)** (14 Points) We define the following linear map

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$\quad b_1 \quad b_2$

i) Calculate an orthonormal basis $F = (f_1, \dots, f_r)$ for $\mathrm{im}(T)$ $= \mathrm{span}\{b_1, b_2\}$

$r = 2$

ii) Check for which $t \in \mathbb{R}$ the vector $v = \begin{pmatrix} 1 \\ t \\ 1 \end{pmatrix}$ is an element in $\mathrm{im}(T)$. Determine the coordinate vector $[v]_F$ in this case.

iii) Find a $w \in \mathbb{R}^3$ with $[w]_F = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

iv) Find a $x \in \mathbb{R}^2$ such that $\|T(x) - b\|$ is minimal, where $b = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$.

(In ii) and iii) the $F$ is the basis of $\mathrm{im}(T)$ you calculated in i)).

$$U = \operatorname{span}\left\{ \underset{b_1}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}, \underset{b_2}{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}} \right\} \subset \mathbb{R}^3$$

iii) Determine a basis for $U^\perp$.

$$U^\perp = \{ x \in \mathbb{R}^3 \mid x \cdot b_1 = x \cdot b_2 = 0 \}$$

$$= \operatorname{Ker}\begin{pmatrix} - b_1 - \\ - b_2 - \end{pmatrix}$$

$$= \operatorname{Ker}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \end{pmatrix}$$

iv) Find a linear map $G : \mathbb{R}^2 \to \mathbb{R}^3$ with $\ker(G) = \{\underset{=\begin{pmatrix} 0 \\ 0 \end{pmatrix}}{0}\}$ and $\operatorname{im}(G) = U$.

$\mathbb{R}^2 \supset \operatorname{Ker}(G)$

$$G : \mathbb{R}^2 \to \mathbb{R}^3$$

$$x \mapsto \begin{array}{c} \\ 3 \\ \end{array}\underbrace{\overbrace{\begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 0 \end{pmatrix}}^{2}} x$$

$$\operatorname{im}(G) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right\} = U$$

$\operatorname{Ker}(G) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ since lin. indep.

$$\underbrace{\lambda_1 \begin{pmatrix}1\\0\\-1\end{pmatrix} + \lambda_2 \begin{pmatrix}1\\2\\0\end{pmatrix}} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\underbrace{\begin{pmatrix}1 & 1\\0 & 2\\-1 & 0\end{pmatrix}}_{A}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\iff \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} \in \ker(A)$$

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A is invertible, B is not invertible

ii) Calculate the matrix $BA$ and decide if $BA^n$ is invertible for any integer $n \geq 1$.

$A^n = \underbrace{A \cdot A \cdots \cdot A}$ is also invertible for all $n \geq 1$

$A^{-1}$ is invertible $\Rightarrow$ $(A^n)^{-1}$ invertible

Is $BA^n$ invertible?

No. If $BA^n$ would be invertible

then $BA^n (A^n)^{-1} = B$ would be invertible

Is

$$\text{ii)}\ U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid \overbrace{2x_1 = x_1 + x_2}^{x_1 - x_2 = 0} \right\}.$$ a subspace?

$\overset{?}{=}$ $\ker(F)$

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto (x_1 - x_2) = (1\ \ {-1}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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$$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid x_1 + x_2 = 0 \text{ and } x_3 = x_1 - x_2 \right\}$$

$-x_1 + x_2 + x_3 = 0$

$= \ker(G)$

$$G : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ -x_1 + x_2 + x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

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$$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 \cdot x_2 = 0 \right\}$$

$H : \mathbb{R}^2 \to \mathbb{R}^1$ ⟵ Not lin. map!

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto (x_1 : x_2)$$

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Sol. for 1) iii) 2022:

Why $\operatorname{Ker}(C) \subset \operatorname{Ker}(C^n)$

If $x \in$ then $Cx = 0$.

Then $\underbrace{C^n x}_{(C \cdots (C x))} = C^{n-1}(Cx) = 0$

$\Longrightarrow x \in \operatorname{Ker}(C^n)$

$$(A \cdot B) \cdot C \overset{\text{Associative}}{=} A \cdot (B \cdot C)$$

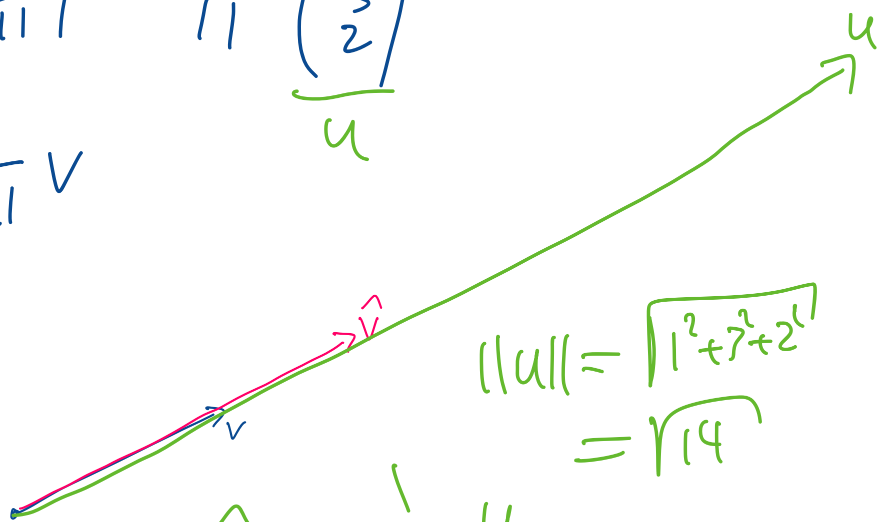
$$\neq$$

$$B(AC)$$

$$V = \begin{pmatrix} \frac{1}{11} \\ \frac{3}{11} \\ \frac{2}{11} \end{pmatrix} = \frac{1}{11} \underbrace{\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}}_{u}$$

$$\|V\| = \sqrt{\left(\frac{1}{11}\right)^2 + \left(\frac{3}{11}\right)^2 + \left(\frac{2}{11}\right)^2}$$

$$\hat{V} = \frac{1}{\|V\|} V$$




$$\|u\| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$$

$$\hat{u} = \frac{1}{\|u\|} u = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \hat{V}$$