Linear Algebra I - Final exam Solution

Nagoya University, G30 Program

Fall 2023

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1) (12 Points) Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -3 \\ -2 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 & -1 \\ -4 & 1 & -1 \\ -3 & 1 & 0 \end{pmatrix}$.

(i) Determine whether or not the matrices A, B are invertible and, if they are, compute their inverses.

(ii) Calculate $C = BA$ and find all vectors $x \in \mathbb{R}^3$ with $Cx = x$.

(iii) Give a basis for $\ker(A^n B)$ for all $n \geq 1$.

2) (14 Points) We define the subspace $U = \text{span}\{u_1, u_2, u_3, u_4\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}, \quad u_3 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix}.$$

(i) Determine a basis $B = (b_1, \dots, b_m)$ of U and calculate its dimension.

(ii) Calculate the coordinate vectors $[u_1]_B$, $[u_2]_B$,$[u_3]_B$, and $[u_4]_B$, where B is the basis from (i).

(iii) Determine an orthonormal basis $F = (f_1, \dots, f_m)$ of U.

(iv) Calculate the orthogonal projection $P_U(b)$ of $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and decide if b is an element in U.

3) (12 Points) Let $u, v \in \mathbb{R}^3$ be two arbitrary <u>non-zero</u> vectors. Which of the following sets are subspaces? Justify your answers.

(i) $U_1 = \left\{ x \in \mathbb{R}^3 \mid x \bullet u = u \bullet u \right\}$.

(ii) $U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid x_1 - x_2 = 3x_3 + x_2 \quad \text{and} \quad x_2 = x_1 - x_3 \right\}$.

(iii) $U_3 = \left\{ x \in \mathbb{R}^3 \mid x \neq u \right\}$.

(iv) $U_4 = \text{span}\{u, v\} \cup \text{span}\{u + v, u - v\}$.

4) (12 Points) Assume we have the following data points

i	1	2	3
x_i	1	2	3
y_i	0	2	3

(i) Find the line of best fit for the above data, i.e. find $a, b \in \mathbb{R}$ such that the function $l(x) = ax + b$ minimizes the sum of squares $\sum_{i=1}^{3} (l(x_i) - y_i)^2$.

(ii) We define the following linear map

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ 2x_1 + x_2 \\ 3x_1 + x_2 \end{pmatrix}$$

and set $V = \text{im}(F)$. Determine the orthogonal projection $P_V(y)$ of $y = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$ onto V.

(iii) Give a basis B of V and determine $[P_V(y)]_B$, where V and y are the same as in (ii).

1) (12 Points) Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -3 \\ -2 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 & -1 \\ -4 & 1 & -1 \\ -3 & 1 & 0 \end{pmatrix}$.

(i) Determine whether or not the matrices A, B are invertible and, if they are, compute their inverses.

(ii) Calculate $C = BA$ and find all vectors $x \in \mathbb{R}^3$ with $Cx = x$.

(iii) Give a basis for $\ker(A^n B)$ for all $n \geq 1$.

(i) We calculate $\mathrm{rref}(A \mid I_3)$ and $\mathrm{rref}(B \mid I_3)$:

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & -3 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -4 & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & -1 \end{array}\right) = (I_3 \mid A^{-1})$$

$\Longrightarrow$ A is invertible with inverse $A^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ -4 & 1 & -1 \\ -2 & 0 & -1 \end{pmatrix}$.

$$(B \mid I_3) = \left(\begin{array}{ccc|ccc} -1 & 0 & -1 & 1 & 0 & 0 \\ -4 & 1 & -1 & 0 & 1 & 0 \\ -3 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 3 & -4 & 1 & 0 \\ 0 & 1 & 3 & 0 & -3 & 1 \end{array}\right)$$

$$\sim \left(\overbrace{\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}}^{\mathrm{rref}(B)} \left| \begin{array}{ccc} -1 & 0 & 0 \\ -4 & 1 & 0 \\ 4 & -4 & 1 \end{array}\right.\right)$$

$\Longrightarrow$ B is not invertible since $\mathrm{rref}(B) \neq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(ii) We have

$$C = BA = \begin{pmatrix} -1 & 0 & -1 \\ -4 & 1 & -1 \\ -3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -3 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}.$$

Since $Cx = \overset{I_3 x}{\tilde{x}}$ is equivalent to $(C - I_3)x = 0$

we need to calculate $\ker(C - I_3)$.

$$C - I_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{rref}(C - I_3)$$

$\Rightarrow$ The solutions $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ of $Cx = x$ are given by

$$\begin{aligned} x_1 &= t_1 - t_2 \\ x_2 &= t_1 \\ x_3 &= t_2 \end{aligned} \qquad \text{for } t_1, t_2 \in \mathbb{R}.$$

(They are $\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$).

(iii) Since A invertible $\Rightarrow A^n$ invertible $\forall n \geq 1$

$$\Rightarrow \operatorname{Ker}(A^n) = \{0\}$$

we have that $A^n y = 0$ if and only if $y=0$.

Therefore $A^n Bx = 0$ if and only if $Bx = 0$.

$$\Rightarrow \operatorname{Ker}(A^n B) = \operatorname{Ker}(B).$$

By (i) we have $\operatorname{rref}(B) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

and therefore the solutions of $Bx = 0$ with $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

are $\begin{array}{l} x_1 = -t \\ x_2 = -3t \\ x_3 = t \end{array}$ for $t \in \mathbb{R}$, i.e. $x = t\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$.

$$\Rightarrow \operatorname{Ker}(A^n B) = \operatorname{Ker}(B) = \operatorname{span}\left\{\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}\right\}$$

$\Rightarrow \left(\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}\right)$ is a basis of $\operatorname{Ker}(A^n B)$ $\forall n \geq 1$.

2) (14 Points) We define the subspace $U = \text{span}\{u_1, u_2, u_3, u_4\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}, \quad u_3 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix}.$$

(i) Determine a basis $B = (b_1, \ldots, b_m)$ of U and calculate its dimension.

(ii) Calculate the coordinate vectors $[u_1]_B$, $[u_2]_B$,$[u_3]_B$, and $[u_4]_B$, where B is the basis from (i).

(iii) Determine an orthonormal basis $F = (f_1, \ldots, f_m)$ of U.

(iv) Calculate the orthogonal projection $P_U(b)$ of $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and decide if b is an element in U.

(i) We calculate $\text{rref}\begin{pmatrix} | & | & | & | \\ u_1 & u_2 & u_3 & u_4 \\ | & | & | & | \end{pmatrix}$:

$$\begin{pmatrix} | & | & | & | \\ u_1 & u_2 & u_3 & u_4 \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & -1 \\ -1 & -3 & 3 & 5 \\ 1 & 3 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 & 1 & -1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Columns with pivot elements.

$$\Rightarrow \quad B = (\overset{b_1}{\overset{\parallel}{u_1}}, \overset{b_2}{\overset{\parallel}{u_3}}) \text{ is a basis of } U.$$

$$\Rightarrow \dim(U) = 2.$$

(ii) From (i) we get

$$[u_1]_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [u_2]_B = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad [u_3]_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad [u_4] = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

(iii) We use the GSA to construct (f_1, f_2) from $(b_1, b_2) = (u_1, u_3)$:

$$f_1 = \frac{1}{\|b_1\|} b_1 = \underline{\underline{\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}}$$

$$f \cdot b_2 = \frac{1}{\sqrt{2}} \overbrace{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}}^{-2}$$

$$w_2 = b_2 - (f_1 \cdot b_2) f_1 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} + \frac{2}{\sqrt{2}\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}.$$

$$f_2 = \frac{1}{\|w_2\|} w_2 = \frac{1}{\sqrt{9}} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \underline{\underline{\frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}}}.$$

(iv) Since (f_1, f_2) is an ONB of U, we have $P_U(b) = (f_1 \cdot b) f_1 + (f_2 \cdot b) f_2$

$$= \left(\frac{1}{\sqrt{2}} \overbrace{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}^{1} \right) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \left(\frac{1}{3} \overbrace{\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}^{9} \right) \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{3}{2} \\ \frac{5}{2} \end{pmatrix} = \underline{\underline{\frac{1}{2} \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}}}.$$

Therefore $P_U(b) \neq b$ and thus $\underline{\underline{b \notin U}}$. (because $P_U(x) = x$ for all $x \in U$)

3) (12 Points) Let $u, v \in \mathbb{R}^3$ be two arbitrary <u>non-zero</u> vectors. Which of the following sets are subspaces? Justify your answers.

(i) $U_1 = \{x \in \mathbb{R}^3 \mid x \bullet u = u \bullet u\}$.

(ii) $U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid \underbrace{x_1 - x_2 = 3x_3 + x_2}_{x_1 - 2x_2 - 3x_3 = 0} \text{ and } \underbrace{x_2 = x_1 - x_3}_{x_1 - x_2 - x_3 = 0} \right\}$.

(iii) $U_3 = \{x \in \mathbb{R}^3 \mid x \neq u\}$.

(iv) $U_4 = \operatorname{span}\{u, v\} \cup \operatorname{span}\{u + v, u - v\}$.

(i) If $u \neq 0$ we have $u \bullet u \neq 0$. But since $0 \bullet u = 0$ we would get $0 \notin U_1 \Rightarrow U_1$ is not a subspace.

(ii) Define the linear map

$$F: \mathbb{R}^3 \to \mathbb{R}^2$$
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 - 2x_2 - 3x_3 \\ x_1 - x_2 - x_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & -3 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Then $U_2 = \operatorname{Ker}(F)$ and therefore U_2 is a subspace.

(iii) If $u \neq 0$ we have $\frac{1}{2}u \neq u$. Therefore $\frac{1}{2}u \in U_3$. But $2 \cdot \left(\frac{1}{2}u\right) = u \notin U_3$ and therefore U_3 is not a subspace.

(iv) Since $u = \frac{1}{2}(u+v) + \frac{1}{2}(u-v)$

$v = \frac{1}{2}(u+v) - \frac{1}{2}(u-v)$

we have $u, v \in \operatorname{span}\{u+v, u-v\}$.

But also $u+v, u-v \in \operatorname{span}\{u, v\}$

$\Longrightarrow \operatorname{span}\{u, v\} = \operatorname{span}\{u+v, u-v\}$.

$\Longrightarrow U_4 = \operatorname{span}\{u, v\} \Rightarrow U_4$ is a subspace.

4) (12 Points) Assume we have the following data points

i	1	2	3
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y_i	0	2	3

(i) Find the line of best fit for the above data, i.e. find $a, b \in \mathbb{R}$ such that the function $l(x) = ax + b$ minimizes the sum of squares $\sum_{i=1}^{3}(l(x_i) - y_i)^2$.

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and set $V = \mathrm{im}(F)$. Determine the orthogonal projection $P_V(y)$ of $y = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$ onto V.

(iii) Give a basis B of V and determine $[P_V(y)]_B$, where V and y are the same as in (ii).

(The Key in this Exercise is to understand that (i) solves (ii) & (iii))

(i) We need to find $\begin{pmatrix} a \\ b \end{pmatrix}$ such that $\| A\begin{pmatrix} a \\ b \end{pmatrix} - y \|$ is minimal, where

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}.$$

For this we need to solve the normal equation

$$A^T A \begin{pmatrix} a \\ b \end{pmatrix} = A^T y.$$

We have

$$A^T A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix}$$

and

$$A^T y = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ 5 \end{pmatrix}$$

$\rightsquigarrow$ Want to solve $\begin{pmatrix}14 & 6\\ 6 & 3\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}13\\ 5\end{pmatrix}$

$$\rightsquigarrow \frac{1}{6}\left(\begin{array}{cc|c}14 & 6 & 13\\ 6 & 3 & 5\end{array}\right) \sim \;(-14)\left(\begin{array}{cc|c}14 & 6 & 13\\ 1 & \frac{1}{2} & \frac{5}{6}\end{array}\right)$$

$$\underbrace{13}_{\frac{39}{3}} - \underbrace{14\cdot\frac{5}{6}}_{\frac{35}{3}} = \frac{4}{3}$$

$$\sim (-1)\left(\tfrac{1}{2}\right)\left(\begin{array}{cc|c}0 & -1 & \frac{4}{3}\\ 1 & \frac{1}{2} & \frac{5}{6}\end{array}\right) \sim \left(\begin{array}{cc|c}1 & 0 & \frac{3}{2}\\ 0 & 1 & -\frac{4}{3}\end{array}\right)$$

$\Rightarrow \quad a=\frac{3}{2}$ and $b=-\frac{4}{3}$

Best fitting line: $\ell(x)=\frac{3}{2}x-\frac{4}{3}$.

(ii) Since $[F]=A$ from (i) we get

$$P_V(b)=A\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}1 & 1\\ 2 & 1\\ 3 & 1\end{pmatrix}\begin{pmatrix}\frac{3}{2}\\ -\frac{4}{3}\end{pmatrix}=\begin{pmatrix}\frac{3}{2}-\frac{4}{3}\\ 3-\frac{4}{3}\\ \frac{9}{2}-\frac{4}{3}\end{pmatrix}$$

$$\underbrace{\frac{3}{2}}_{\frac{9}{6}}-\underbrace{\frac{4}{3}}_{\frac{8}{6}}=\frac{1}{6}$$

$$\underbrace{3}_{\frac{9}{3}}-\frac{4}{3}=\frac{5}{3}$$

$$\underbrace{\frac{9}{2}}_{\frac{27}{6}}-\underbrace{\frac{4}{3}}_{\frac{8}{6}}=\frac{19}{6}$$

$$=\begin{pmatrix}\frac{1}{6}\\ \frac{5}{3}\\ \frac{19}{6}\end{pmatrix}=\frac{1}{6}\begin{pmatrix}1\\ 10\\ 19\end{pmatrix}$$

(iii) We can take $B = \left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$, since $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are lin. independent and they span $\mathrm{im}(F)$.

Since $P_V(b) = A \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

we have

$$[P_V(b)]_B = \begin{pmatrix} a \\ b \end{pmatrix} = \underline{\underline{\begin{pmatrix} \frac{3}{2} \\ -\frac{4}{3} \end{pmatrix}}}.$$