1) (12 Points) Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -3 \\ -2 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 & -1 \\ -4 & 1 & -1 \\ -3 & 1 & 0 \end{pmatrix}$.

(i) Determine whether or not the matrices A, B are invertible and, if they are, compute their inverses.

(ii) Calculate $C = BA$ and find all vectors $x \in \mathbb{R}^3$ with $Cx = x$.

(iii) Give a basis for $\ker(A^n B)$ for all $n \geq 1$.

2) (14 Points) We define the subspace $U = \operatorname{span}\{u_1, u_2, u_3, u_4\} \subset \mathbb{R}^3$, where

$$u_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}, \quad u_3 = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix}.$$

(i) Determine a basis $B = (b_1, \dots, b_m)$ of U and calculate its dimension.

(ii) Calculate the coordinate vectors $[u_1]_B$, $[u_2]_B$,$[u_3]_B$, and $[u_4]_B$, where B is the basis from (i).

(iii) Determine an orthonormal basis $F = (f_1, \dots, f_m)$ of U.

(iv) Calculate the orthogonal projection $P_U(b)$ of $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and decide if b is an element in U.

3) (12 Points) Let $u, v \in \mathbb{R}^3$ be two arbitrary <u>non-zero</u> vectors. Which of the following sets are subspaces? Justify your answers.

(i) $U_1 = \left\{ x \in \mathbb{R}^3 \mid x \bullet u = u \bullet u \right\}$.

(ii) $U_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \mid x_1 - x_2 = 3x_3 + x_2 \quad \text{and} \quad x_2 = x_1 - x_3 \right\}$.

(iii) $U_3 = \left\{ x \in \mathbb{R}^3 \mid x \neq u \right\}$.

(iv) $U_4 = \operatorname{span}\{u, v\} \cup \operatorname{span}\{u + v, u - v\}$.

4) (12 Points) Assume we have the following data points

i	1	2	3
x_i	1	2	3
y_i	0	2	3

(i) Find the line of best fit for the above data, i.e. find $a, b \in \mathbb{R}$ such that the function $l(x) = ax + b$ minimizes the sum of squares $\sum_{i=1}^{3} (l(x_i) - y_i)^2$.

(ii) We define the following linear map

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ 2x_1 + x_2 \\ 3x_1 + x_2 \end{pmatrix}$$

and set $V = \operatorname{im}(F)$. Determine the orthogonal projection $P_V(y)$ of $y = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$ onto V.

(iii) Give a basis B of V and determine $[P_V(y)]_B$, where V and y are the same as in (ii).