Tutorial 7: Review for the midterm exam

Exercise 1. Consider the linear map

$$\begin{aligned} F : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ x &\longmapsto \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 2 \\ 1 & 1 & -5 \end{pmatrix} x\,. \end{aligned}$$

i) Calculate $\mathrm{im}(F)$.

ii) Is F surjective and/or injective?

iii) Find all solutions to $F(x) = 0$.

iv) Find all $x \in \mathbb{R}^3$ such that $v \bullet x = 0$ for all $v \in \mathrm{im}(F)$.

The solutions for Exercises 2-7 below can be found at "Tutorial 6 2019" on the homepage.

Exercise 2. Give an example of a linear system which has

i) exactly one solution.

ii) infinitely many solutions.

iii) no solutions.

Exercise 3. Which of the following matrices are on row-reduced echelon form?

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Exercise 4. Consider the following linear system

$$\begin{cases} x_1 + 4x_2 + 7x_3 + 2x_4 = 1 \\ 2x_1 + 5x_2 + 8x_3 + x_4 = 2 \\ 3x_1 + 6x_2 + 10x_3 + x_4 = 1 \end{cases}.$$

i) Find a matrix $A \in \mathbb{R}^{3\times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

ii) Calculate the row-reduced echelon form of $(A \mid b)$.

iii) Find all the solutions to the linear system.

iv) Calculate the rank of $(A \mid b)$ and A.

Exercise 5. Define for a matrix $A \in \mathbb{R}^{m \times n}$ the linear map

$$\begin{aligned} F_A : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ x &\longmapsto Ax \,. \end{aligned}$$

Choose for A the matrices appearing in Exercise 2 & 3 and decide for each case if F_A is injective and/or surjective and calculate the image of F_A.

Exercise 6.

i) Determine all vectors that are orthogonal to the vector $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

ii) Find a linear map $G : \mathbb{R}^2 \to \mathbb{R}^3$, such that the image of G is given by all the vectors which are orthogonal to v. (i.e. the image gives all the vectors you determined in i)).

Exercise 7. Let $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$ and define the following four functions:

$$\begin{aligned} f_1 : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \mathrm{rot}_x(u) \,, \end{aligned}$$

$$\begin{aligned} f_2 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \begin{pmatrix} u \bullet x \\ (u \bullet u)(x \bullet u) \end{pmatrix} \,, \end{aligned}$$

$$\begin{aligned} f_3 : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &\longmapsto \begin{pmatrix} x_1 + x_2 + x_3 \\ 1 \end{pmatrix} \,, \end{aligned}$$

$$\begin{aligned} f_4 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\longmapsto \begin{pmatrix} 2x_2 - x_1 \\ 2x_1 \\ 4x_1 + x_2 \end{pmatrix} \,. \end{aligned}$$

i) Give a geometric interpretation of the image of f_1. Is f_1 injective and/or surjective?

ii) Which of the above functions f_1, f_2, f_3, f_4 are linear maps? For each one that is linear, determine its matrix.